A Tractable Framework for Analyzing a Class of Nonstationary Markov Models

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Stationary Markov class of models

The mainstream of the literature on economic dynamics builds on stationary Markov models:

- *Markov property* – the optimal value and/or decision functions depend only on current economy's state and are memoryless concerning the history that leads to the current state (state-contingent).
- *Stationarity* – the optimal value and/or decision functions are time-invariant (autonomous).

Some interesting economic models are nonstationary

Conventional stationary Markov framework may be too restrictive for some interesting applications:

- unbalanced growth patterns in the data;
- different kinds of technological progress that augment factors' productivity,
  King, Plosser and Rebelo (1988);
- an entry into a monetary union;
- nonrecurrent policy regime switches;
- deterministic seasonals;
- changes in the consumer's tastes and habits.

In such models, the optimal value and/or decision functions are not time-invariant but change from one period to another.

A class of nonstationary Markov models

- In this paper, we relax the restriction of stationarity, and we consider a class of *infinite-horizon*, nonlinear dynamic economic models with changing over time:
  - preferences;
  - technology;
  - laws of motion for exogenous shocks.
- They change either deterministically or stochastically, according to a Markov process, or both.
- A distinctive feature of our analysis is that we allow for **Markov processes with time-varying transition probabilities**.
- For example, $z_{t+1} = \rho_t z_t + \sigma_t \varepsilon_{t+1}$, where $\rho_t < 1$, $\sigma_t \in (0, \infty)$ are given at $t = 0$ for all $t \geq 0$ and $\varepsilon_{t+1} \sim \mathcal{N}(0, 1)$.
- The studied models are nonstationary in the sense that the optimal value and consumption functions are of the form $V_t(k, z)$ and $c_t = C_t(k, z)$, respectively, i.e.,
  - preserve Markov (state-contingent) structure;
  - but are not time-invariant.

Why cannot we solve a nonstationary model with the usual methods

A stationary growth model (dynamic-programming formulation):

$$V(k,z) = \max_{c,k'} \left\{ u(c) + \beta E\left[V\left(k',z'\right)\right]\right\}$$

$$\text{s.t. } k' = (1-\delta)k + zf(k) - c,$$

$$\ln z' = \rho \ln z + \varepsilon', \qquad \varepsilon' \sim \mathcal{N}\left(0,\sigma^2\right).$$

An interior solution satisfies the Euler equation:

$$u'(c) = \beta E\left[u'\left(c'\right)\left(1-\delta+z'f'\left(k'\right)\right)\right].$$

- Conventional value iterative methods: *iterate on Bellman equation until fixed-point* V *is found.*
- Conventional Euler equation methods: *iterate on Euler equation until fixed-point decision function* $k' = K(k,z)$ *is found.*
- However, if u, f, ρ and σ are time-dependent, then $V_t(\cdot) \neq V_{t+1}(\cdot)$ and $K_t(\cdot) \neq K_{t+1}(\cdot)$ and conventional numerical methods, that solve for fixed-point functions V and K, cannot be used.

Balanced growth models

Some nonstationary models that can be converted into stationary models, for example, a class of balanced growth models.

However, the class of balanced growth models is limited:

- King, Plosser and Rebelo (1988) show that the standard growth model is consistent with balanced growth only under the assumption of labor augmenting technological progress and under some additional restrictions on u and f.
- If one deviates from their assumptions, the property of balanced growth does not survive.

We develop a framework for solving nonstationary models that does not rely on the existence of a balanced growth path.

Extended function path framework

- We introduce a simple quantitative framework, called **extended function path** (EFP), for calibrating, solving, simulating and estimating nonstationary Markov models.
- EFP constructs a sequence (path) of a given length T of time-dependent value functions $(V_0, ..., V_T)$ and/or decision functions $(K_0, ..., K_T)$.
- We define a supplementary economy that becomes stationary in some remote period T.

Definition. *(T-period stationary economy).* A T-period stationary economy is the version of our nonstationary economy in which for $t \geq T$, the utility and production functions and the laws of motions for exogenous shocks are time invariant, i.e., $u_t = u$, $f_t = f$ and $\rho_t = \rho$, $\sigma_t = \sigma$ for all $t \geq T$.

The key idea of EFP is to approximate an optimal program in the nonstationary Markov economy with an optimal program in the supplementary T-period stationary economy.

Extended function path (EFP) framework

Step 0. Initialization. Choose some $T \gg \tau$ and construct a T-period stationary economy by setting $u_t = u$, $f_t = f$, $\rho_t = \rho$, $\sigma_t = \sigma$ for all $t \geq T$.

Step 1. Construct a stationary Markov solution,

i.e., find a stationary capital function K satisfying:

$$u'(c) = \beta E\left[u'(c')(1 - \delta + f'(k', \rho z + \sigma \varepsilon'))\right]$$

$$c = (1-\delta)k + f(k, z) - k'$$

$$c' = (1-\delta)k' + f(k', \rho z + \sigma\varepsilon') - k''$$

$$k' = K(k, z) \text{ and } k'' = K(k', \rho z + \sigma \varepsilon').$$

Step 2. Construct a path for capital policy functions $(K_0, \ldots, K_T)$ that matches the terminal condition $K_T \equiv K$ and satisfies for $t = 0, \ldots T-1$:

$$u_t'(c_t) = \beta E_t\left[u_{t+1}'(c_{t+1})(1 - \delta + f'_{t+1}(k_{t+1}, \rho_t z_t + \sigma_t \varepsilon_{t+1}))\right]$$

$$c_t = (1-\delta)k_t + f_t(k_t, z_t) - k_{t+1}$$

$$c_{t+1} = (1-\delta)k_{t+1} + f_{t+1}(k_{t+1}, \rho_t z_t + \sigma_t \varepsilon_{t+1}) - k_{t+2}$$

$$k_{t+1} = K_t(k_t, z_t) \text{ and } k_{t+2} = K_{t+1}(k_{t+1}, \rho_t z_t + \sigma_t \varepsilon_{t+1}).$$

Output: the first τ functions $(K_0, \ldots, K_\tau)$ constitute an approximate solution.

Example of function path constructed by EFP

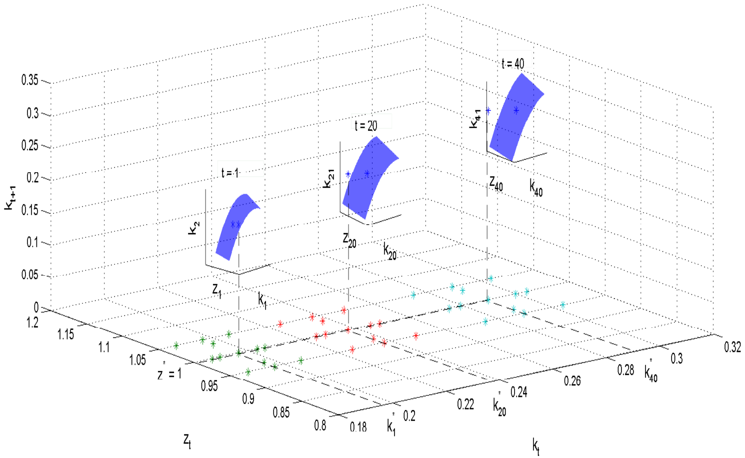


Figure 1. Function path, produced by EFP, for a growth model with technological progress

Extended path versus extended function path

The term *"extended path"* indicates that EFP constructs a path of functions for a larger time horizon T than the number of periods τ for which an approximate solution is actually needed.

- In this respect, EFP is similar to extended path (EP) framework of Fair and Taylor (1983).
- By choosing sufficiently large T, both EFP and EP mitigate the effect of specific terminal condition on the approximation during the initial τ periods.

In turn, the term *"path"* versus *"function path"* highlights the key difference between the EP and EFP methods:

- Fair and Taylor's (1983) EP method constructs a path for variables under the assumption of certainty equivalence.
- EFP method constructs a path for decision functions by approximating expectation functions accurately using accurate deterministic integration methods such as Gauss-Hermite quadrature and monomial methods.

Theoretical foundations of EFP framework

We develop theoretical foundations of the extended function path framework. We prove two theorems:

- *Theorem 1 (existence):* EFP approximations exists, is unique and possess a Markov structure.
- *Theorem 2 (turnpike):* EFP can approximate time-dependent solution to a nonstationary Markov model with an arbitrary degree of precision as the time horizon T increases.

Illustration of turnpike theorem

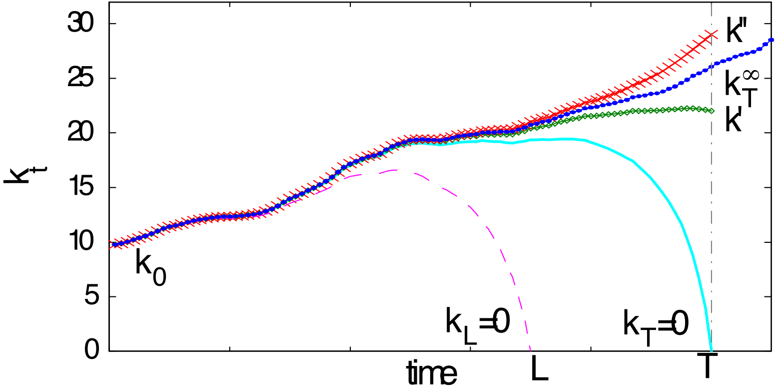


Figure 2. Convergence of the optimal program of the T-period stationary economy

Earlier literature on nonstationary stochastic growth models

- Majumdar and Zilcha (1987), Mitra and Nyarko (1991), and Joshi (1997) study infinite-horizon, nonstationary economies similar to ours without assuming stationarity and Markov structure of the solutions.
- However, this literature is limited to purely theoretical analysis and does not offer practical methods for constructing their nonstationary solutions in applications.
- *Our main contribution:* we distinguish a tractable class of nonstationary models and propose a framework for studying quantitative implications of such models.

Life-cycle models

- EFP is particularly close to numerical methods that construct decision functions in life-cycle models
  e.g., Krueger and Kubler (2004, 2006) and Hasanhodzic and Kotlikoff (2013).
- The decision functions in such models differ from one generation to another, and the sequence of the generation-specific decision functions resembles a function path constructed by EFP.
- The difference is that terminal condition is known in the life-cycle economy while it is unknown in our analysis and must be constructed in the way that ensures the convergence of an EFP approximation to the true nonstationary solution.
- *Our contribution:* for the case of the unknown terminal condition, we show formally that the solution for the first τ periods asymptotically converges to the true solution.

Augmenting state space of economic models

- EFP is close to the literature that incorporates certain kinds of nonstationarity by augmenting the economic models to include additional state variables.
- *Stochastic volatility*; e.g., Bloom (2009), Fernández-Villaverde and Rubio-Ramírez (2010), Fernández-Villaverde et al. (2010) argue that the behavior of real-world economies is affected by degrees of uncertainty.
- *Periodic unanticipated changes in regimes; e.g.*, Davig and Leeper (2009), Farmer, Waggoner and Zha (2011), Foerster, Rubio-Ramírez, Waggoner and Zha (2013).
- *News;* Schmitt-Grohé and Uribe (2012) introduce a quantitative framework that allows for anticipated exogenous shocks of a fixed periodicity and length.
- *The key difference of the EFP framework:* it allows for time dependence of the model itself while the above literature expends the state space of time-invariant models.

Methods constructing a path for variables

- Shooting methods that solve for path $\{k_1, ..., k_T\}$ by using Gauss-Siedel iteration; introduced to economics in Lipton, Poterba, Sachs and Summers (1980).
- A shortcoming of shooting methods is that they tend to produce explosive paths, see Atolia and Buffie (2009).
- Fair and Taylor (1983) introduced an extended path method that can be used to solve economic models with uncertainty.
- However, it becomes highly inaccurate when either volatility and/or the degrees of nonlinearity increase.

Assessing EFP accuracy in a model with balanced growth

- We assess the quality of approximations produced by EFP in the context of a model with balanced growth parameterized by

$$u_t(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \text{and} \quad f_t(k,z) = zk^{\alpha} A_t^{1-\alpha}, \tag{1}$$

  where $\gamma > 0$ and $\alpha \in (0,1)$; $A_t = A_0 g_A^t$ represents a labor augmenting technological progress with an exogenous constant growth rate $g_A \geq 1$.
- Productivity follows

$$\ln z_{t+1} = \rho \ln z_t + \sigma \omega_{t+1}, \qquad \varepsilon_{t+1} \sim \mathcal{N}(0,1), \tag{2}$$

  where $\rho \in (-1,1)$, $\sigma \in (0,\infty)$.
- This version of the model is consistent with balanced growth and can be converted into a stationary model; see King, Plosser and Rebelo (1988). We can solve the stationary model very accurately and use the accurate solution for comparison.

A comparison of four solution methods

We solve the nonstationary growth model using four alternative solution methods:

1. *"Exact solution"* is a very accurate solution to the stationary model with a balanced growth path produced by the conventional Smolyak method;
2. *"EFP solution"* is produced by EFP method that solves a nonstationary model directly;
3. *"Fair and Taylor solution"* is produced under the certainty equivalence assumption for approximating expectation functions;
4. *"Naive solution"* is produced by replacing the nonstationary model with a sequence of stationary models and that solves such models one by one. (The naive method differs from EFP in that it neglects the connection between the decision functions of different periods).

Parameterization

- For all experiments, we fix $\alpha = 0.36$, $\beta = 0.99$, $\delta = 0.025$ and $\rho = 0.95$.
- The remaining parameters are set in the benchmark case at $\gamma = 5$, $\sigma_\varepsilon = 0.03$, $g_A = 1.01$ and $T = 200$, and we vary these parameters across experiments.
- For all simulations, we use the same initial condition and the same sequence of productivity shocks.
- Our code is written in MATLAB 2013a, and we use a desktop computer with Intel(R) Core(TM) i7-2600 CPU (3.40 GHz) with RAM 12GB.
- The running times for EFP can be reduced considerably if we use parallelization (our iteration, which is in line with Gauss-Jacobi method, is naturally parallelizable).

Critical role of expectations in the accuracy of solutions

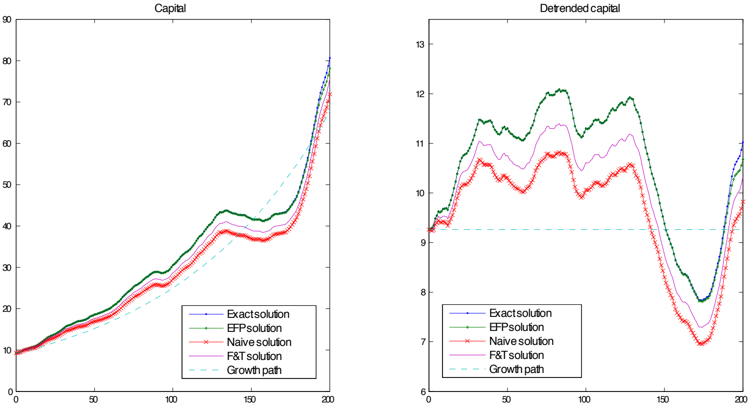


Figure 3. Comparison of the solution methods for the test model with balanced growth

Table 1: comparison of four solution methods

	Fair-Taylor (1983) method, $\tau = 1$		Naive method	EFP method $\tau = 200$		
Terminal condition	Steady state	Steady state	-	Balanced growth	T-period stationary	
T	200	400	200	200	200	400
Maximum errors across t periods in $\log_{10}$ units						
$t \in [0, 50]$	-1.29	-1.29	-1.04	-6.82	-6.01	-6.42
$t \in [0, 100]$	-1.18	-1.18	-0.92	-6.68	-4.39	-5.99
$t \in [0, 150]$	-1.14	-1.14	-0.89	-6.66	-2.89	-5.98
$t \in [0, 175]$	-1.14	-1.13	-0.89	-6.66	-2.10	-5.98
$t \in [0, 200]$	-1.14	-1.13	-0.89	-6.66	-1.45	-5.92
Running time, in seconds						
Solution	1.2(+4)	6.1(+4)	28.9	104.9	99.1	225.9
Simulation	-	-	2.6	2.6	2.8	5.7
Total	1.2(+4)	6.1(+4)	31.5	107.6	101.9	231.6

Application 1: model with capital augmenting technological progress (cont.)

- We assume a constant elasticity of substitution (CES) production function, and we allow for both labor and capital augmenting technological progresses,

$$F(k_t, \ell_t) = \left[\alpha (A_{k,t} k_t)^v + (1-\alpha)(A_{\ell,t}\ell_t)^v\right]^{1/v},$$

  where $A_{k,t} = A_{k,0} g_{A_k}^t$; $A_{\ell,t} = A_{\ell,0} g_{A_\ell}^t$; $v \leq 1$; $\alpha \in (0,1)$; g_{A_k} and g_{A_ℓ} are the rates of capital and labour augmenting technological progresses, respectively.
- Labor is supplied inelastically. Let $\ell_t = 1$ for all t. The corresponding production function by $f(k_t) \equiv F(k_t, 1)$.
- The model with capital augmenting technological progress does not satisfy the assumptions in King, Plosser and Rebelo (1988) and does not admit a balanced growth path.

Capital versus labor augmenting technological progress

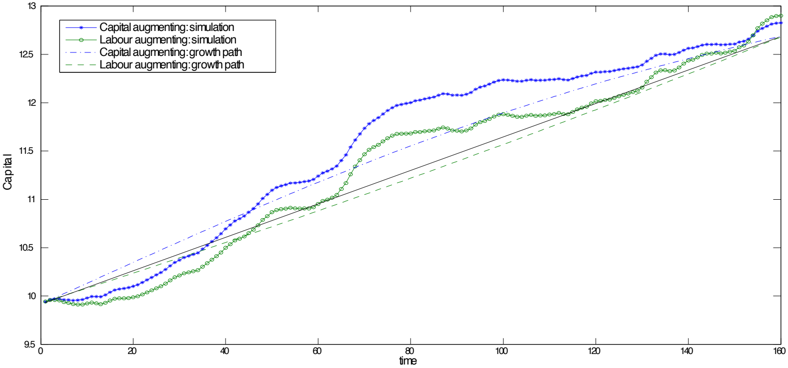


Figure 4: Technological progress in the model with the CES production

Application 2a: A nonstationary model with a parameter shift

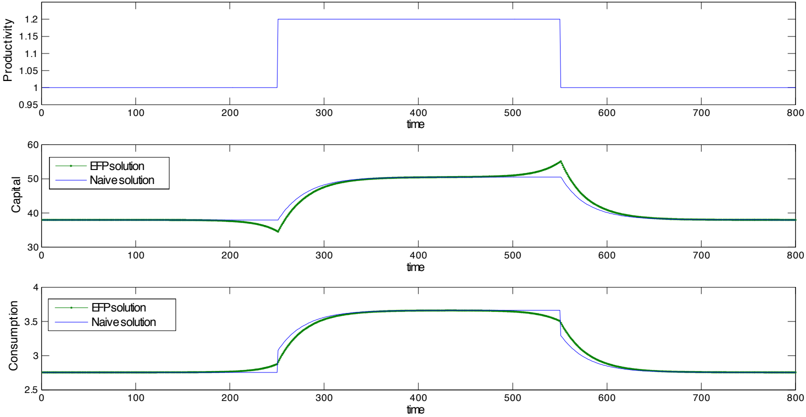


Figure 5. Anticipated versus unanticipated technology shocks

Application 2b: A model with seasonal changes

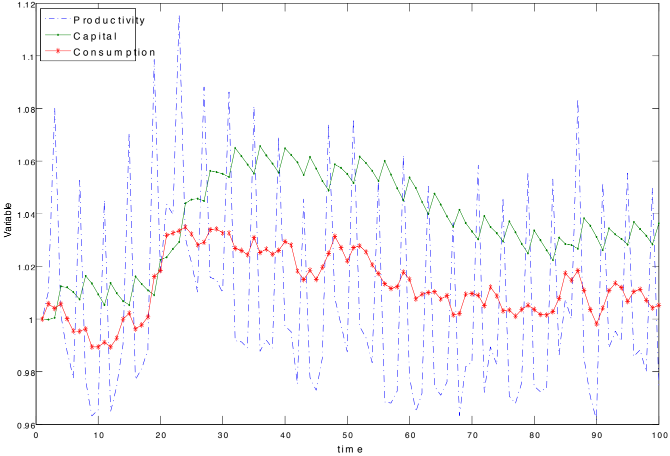
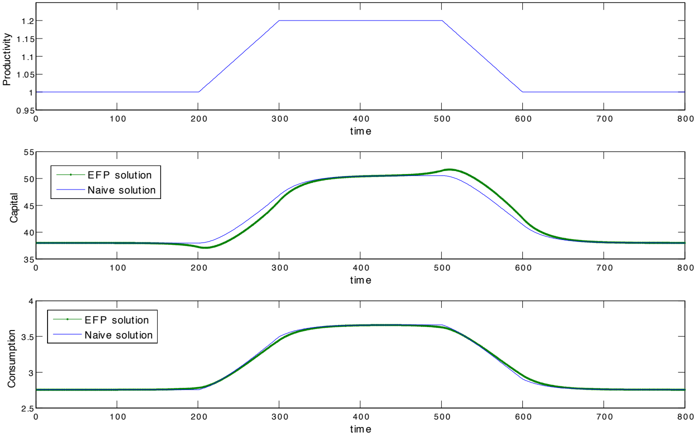


Figure 6. Seasonality

Application 3a: A nonstationary model with a parameter drift



Application 3b: Diminishing volatility

- We show how to use EFP to study a model in which the volatility has both a stochastic and deterministic components.
- We specifically consider the standard neoclassical stochastic growth model, modified to include a diminishing volatility of the productivity shock:

$$\ln z_t = \rho \ln z_{t-1} + \sigma_t \varepsilon_t, \qquad \sigma_t = \frac{B}{t^{\rho_\sigma}}, \quad \varepsilon_t \sim \mathcal{N}(0,1), \qquad (3)$$

  – B = a scaling parameter;
  – ρ_σ is a parameter that governs the volatility of z_t.

- The standard deviation of the productivity shock $B\sigma / t^{\rho_\sigma}$ decreases over time, reaching zero in the limit, $\lim_{t\to\infty} \frac{B\sigma}{t^{\rho_\sigma}} = 0$.

Application 3: Diminishing volatility (cont.)

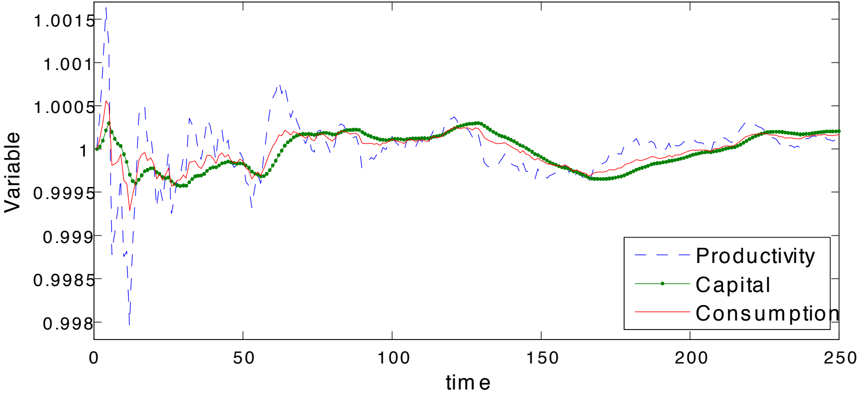


Figure 8. Diminishing volatility

Application 4: Calibrating a growth model with a parameter drift to unbalanced U.S. data

We extend the benchmark model to include time-varying depreciation rate of capital,

$$\max_{\{c_t, k_{t+1}\}_{t=0,\ldots,\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t. } c_t + k_{t+1} = A_t z_t k_t^{\alpha} + (1 - d_t \delta_t) k_t,$$

$$\ln \delta_t = \rho_\delta \ln \delta_{t-1} + \varepsilon_{\delta,t}, \qquad \varepsilon_{\delta,t} \sim \mathcal{N}\left(0, \sigma^2_{\varepsilon_d}\right),$$

$$\ln z_t = \rho_z \ln z_{t-1} + \varepsilon_{z,t}, \qquad \varepsilon_{z,t} \sim \mathcal{N}\left(0, \sigma^2_{\varepsilon_z}\right),$$

$d_t \delta_t$ = a time-varying depreciation rate; d_t = a trend component of depreciation, $d_t = d_0 g_d^t$; δ_t = a stochastic shock to depreciation.

Fitted time series

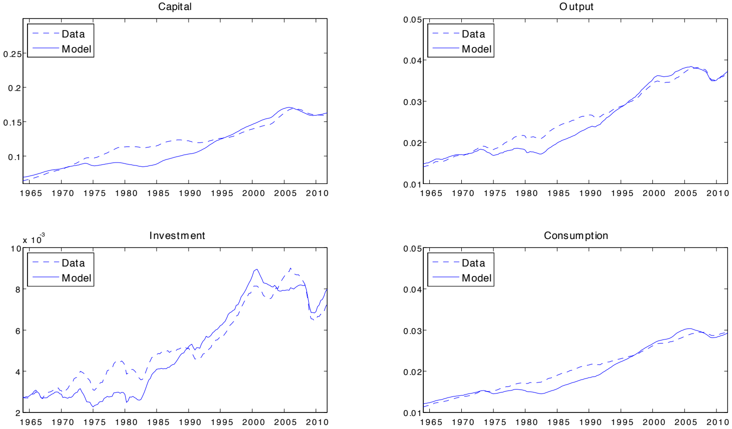


Figure 9. Matching nonstationary macroeconomic data on the U.S. economy

Conclusion

- Stationary Markov class of models is a dominant framework in recent economic literature.
- A shortcoming of this framework is that it generally restricts the parameters of economic models to be constant, and it restricts the behavior patterns to be time invariant.
- In this paper, we construct a more flexible class of nonstationary Markov models that allows for time-varying structural parameters and decision functions.
- We propose EFP framework for solving, calibrating, simulating and estimating of parameters in such models. EFP enables us to analyze economic models that do not admit stationary Markov equilibria and that cannot be studied with conventional solution methods.
- Literally, EFP makes it possible to construct a unique historical path of real-world economies.