Entropy-Based Approximations of Dynamic Equilibrium Models: A Unified Theory of Risk-Adjusted Linearizations

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Abstract

We propose a simple risk-adjusted linear approximation to solve a large class of dynamic models with time-varying and non-Gaussian risk. Our approach generalizes lognormal affine approximations commonly used in the macro-finance literature and can be seen as a first-order perturbation around the risky steady state. Thus, we provide a mathematical foundation for approximation methods that remained so far heuristic, and we unify coexisting theories of risk-adjusted linearizations. Generically, affine approximations are not nested in conventional perturbations of arbitrary order. We offer explicit formulas for approximate equilibrium quantities and asset prices and conditions for their local existence and uniqueness. We apply this technique to models featuring Campbell-Cochrane habits, recursive preferences and time-varying disaster risk. In these examples the proposed affine approximation performs similarly to global solution methods; risk pricing is accurate at all investment horizons, thereby capturing the main properties of investors' marginal utility of wealth and measures of welfare costs of fluctuations.

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1. Introduction

Variation in risk sensitivity is a central ingredient in modern dynamic equilibrium models. It is necessary to capture asset pricing facts and to study the real effects of macroeconomic uncertainty.

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Leading examples include time-varying risk aversion, risk-sensitive preferences, stochastic volatility, variable disaster risk and intertemporal portfolio choice.[2] These models present a challenge for extant solution techniques. Projection methods are accurate but computationally intensive and offer limited analytic insight. Higher-order perturbations around the deterministic steady state have similar disadvantages and can misrepresent the model's implications when non-analytic functions are involved. Finally, the deterministic steady state can be an invalid expansion point.[3]

This paper proposes an affine approximation technique to solve and simulate dynamic stochastic general equilibrium models that includes a time-varying risk adjustment in equilibrium prices and quantities. Affine risk adjustments based on lognormality have been popular in the macro-finance literature at least since Hansen and Singleton (1983), as they facilitate an analytic understanding of the asset pricing implications of the model and the use of fast filtering techniques, yet they remain limited in scope and lack a formal justification based on perturbation theory.[4] Concurrently, a recent literature has been exploring linear approximations around the *risky* steady state, yet their relationship with standard risk-adjusted linearizations has not been explored and a characterization of their exact solution and local stability properties has so far proved elusive.[5]

We make two main contributions. First, we generalize affine approximations common in the macro-finance literature. We extend risk adjustments to non-Gaussian distributed shocks by relying on relative entropy as the measure of dispersion and on the cumulant generating function of shocks, and we ensure consistent risk corrections that are continuous in the model's parameters.[6]

Second, and perhaps most importantly, we provide a unified theory of risk-adjusted linearizations. We show that affine approximations to the policy function are equivalent to its first-order perturbation around the risky steady state, and can therefore be justified on formal grounds based on the implicit function and Taylor theorems. Conversely, risky steady state perturbations can be traced back at least to the 1990s in the finance literature.[7] At the same time, we provide the first complete description of first-order perturbations around the risky steady state by providing explicit

[2]For example, among many others, Campbell and Cochrane (1999); Chamberlain and Wilson (2000); Bansal and Yaron (2004); Binsbergen et al. (2012); Gourio (2012); Rudebusch and Swanson (2012); Fernández-Villaverde et al. (2011); Wachter (2013); Fernández-Villaverde et al. (2015). See also Cochrane (2011) and Bloom (2014).

[3]As in portfolio choice problems in small open economies (see Schmitt-Grohé and Uribe, 2003, and section 3).

[4]Examples of loglinear-lognormal methods applied to asset pricing include Campbell (1993); Jermann (1998); Lettau and Uhlig (2000); Bansal and Yaron (2004); Uhlig (2007); Alvarez et al. (2007); Bekaert et al. (2010); Verdelhan (2010); Dew-Becker (2014); Malkhozov (2014); Backus et al. (2015) among many others.

[5]The risky steady state is the point where agents choose to stay while expecting shocks in the future and when ex-post realized shocks are zero (Coeurdacier et al., 2011; Juillard, 2011). In contrast, the deterministic steady state is the point where agents choose to stay while knowing that future realized shocks are zero.

[6]In particular, extant affine approaches (Malkhozov, 2014, is the most recent formalization of the lognormal affine method) accommodate *dynamic* risk corrections only when the conditional distribution of innovations in state variables is known a priori. In contrast, we accommodate dynamic risk corrections even when risk is *endogenous* in the sense that the distribution of innovations in state variables is known only after the model is solved (e.g., in production economies with habits or in portfolio choice under uncertainty). Therefore, our approximation is continuous at parametrizations at which production economies coincide with endowment economies, a natural consistency requirement.

[7]Coeurdacier et al. (2011) and Juillard (2011) are the first to study explicitly perturbations around the risky steady state; Meyer-Gohde (2016) extends the analysis to perturbations around the ergodic mean. The extant literature does not characterize explicitly the risky steady state and its local properties—a difficulty we bypass by focusing on expectational equations with almost surely positive arguments that grant a connection with relative entropy.

formulas for equilibrium parameters and by characterizing their local existence and uniqueness by generalizing Blanchard and Kahn (1980) conditions. Generically, our approximation is not nested in deterministic steady state perturbations of arbitrary order—which may not even be well-defined in examples in which risk-adjusted perturbations are—so the mathematics involved is novel.[8]

Affine approximations are particularly appealing by their ability to provide analytic solutions to price easily complex dividend processes (such as those arising from a production economy) and that facilitate an intuitive understanding of the macroeconomic forces that drive asset prices, and hence investors' marginal utility of wealth and welfare. We provide pricing formulas in the affine class for equilibrium term structures—and hence for all related claims, including wealth portfolios and welfare costs of fluctuations (see Alvarez and Jermann, 2004; Lopez, 2014)—as well as for some major diagnostic decompositions of the asset pricing properties of a model, including Hansen and Jagannathan (1991) and Backus, Chernov, and Zin (2014) entropy-based bounds, Alvarez and Jermann (2005) and Hansen and Scheinkman (2009) decompositions, and Borovicka and Hansen (2014) risk-exposure and risk-price elasticities.

To test the accuracy of our method in a challenging context we consider general equilibrium models with Campbell and Cochrane (1999) habits and models with recursive utility and a risk of rare disasters. Projection methods are notoriously required to find the global solution under nonlinear habits; rare disasters are main examples of non-Gaussian exogenous shocks. We emphasize the importance of an accurate solution for the term structures of zero-coupon equities and bonds, as they are the basis for pricing more complex claims and they capture the diagnostic measures that sum up a model's implications for investors' marginal utilities and, more generally, asset pricing kernels.

Throughout the examples, our approximation outperforms alternative approximation schemes in its representation of equilibrium quantities and asset prices. We first test the performance of our approximation procedure in the endowment economies of Campbell and Cochrane (1999) and Wachter (2013). Our approximation is accurate in solving for risk premia and volatilities of equities and bonds at both short and long durations, and produces continuous disaster-risk corrections.

We then turn to production economies. The models of Jermann (1998)—with habit formation and capital accumulation—and Lopez, Lopez-Salido, and Vazquez-Grande (2015)—with habit formation and nominal rigidities—are appropriate for testing the accuracy of our solution in an environment where consumption risk is endogenous. Finally, we apply our approximation method to the production economy of Gourio (2012) with rare disasters with time-varying severity. In these applications the full nonlinear solution is computationally expensive, while our generalized affine approximation yields a fast and tractable solution with good accuracy.

The rest of the paper is structured as follows. Section 2 describes our affine approximation heuristically. Section 3 illustrates it in simple examples. Section 4 presents formal results motivating our method. Section 5 inspects approximate risk pricing formulas. Section 6 applies the method to main examples in the literature. Section 7 concludes. Proofs are in the Appendix.

[8]Malkhozov (2014) suggests that affine approximations are second-order perturbations around the deterministic steady state in which dynamic second-order terms are disregarded, yet we show that the connection to perturbations can be made exact. Proposition 2 describes further the relationship between affine approximations and perturbations around the deterministic steady state.

2. Approximation method: Heuristic algorithm

2.1. A basic example

Suppose we want to price the equilibrium log risk-free rate (r) described by the Euler equation:

$$0 = \ln E_t e^{\ln(\beta) - \gamma \Delta c_{t+1} - \gamma \Delta s_{t+1} + r_t}$$

for some parameters $\beta \in (0, 1)$ and $\gamma > 0$, and by the law of motion of the state vector:

$$\hat{s}_{t+1} = \phi \hat{s}_t + \Lambda(\hat{s}_t) \sigma \varepsilon_{t+1}$$

for some nonlinear function $\Lambda(\hat{s}_t)$ and consumption $c_{t+1} = \mu + c_t + \sigma \varepsilon_{t+1}$, with $\varepsilon_t \sim Niid(0, 1)$.

The idea of affine approximations is to conjecture an affine solution in the state $r_t = \widetilde{r} + \widetilde{\psi}_r \hat{s}_t$ and to rewrite the Euler equation using the normality of shocks as:

$$\begin{aligned} 0 &= \ln(\beta e^{-\gamma\mu}) + \gamma(1 - \phi)\hat{s}_t + \widetilde{r} + \widetilde{\psi}_r \hat{s}_t + \frac{1}{2} var_t \left(-\gamma\sigma\varepsilon_{t+1} - \gamma \Delta s_{t+1}\right) \\ &= \ln(\beta e^{-\gamma\mu}) + \gamma(1 - \phi)\hat{s}_t + \widetilde{r} + \widetilde{\psi}_r \hat{s}_t + \gamma^2 [1 + \Lambda(\hat{s}_t)]^2 \frac{\sigma^2}{2} \end{aligned}$$

Linearizing any remaining nonlinearities and matching coefficients would identify the solution as:

$$\widetilde{r} = -\ln(\beta e^{-\gamma\mu}) - \gamma^2 [1 + \Lambda(0)]^2 \frac{\sigma^2}{2}, \qquad \widetilde{\psi}_r = -\gamma(1 - \phi) - \gamma^2 [1 + \Lambda(0)] \Lambda_1(0) \sigma^2$$

where the last term in both coefficients represent risk corrections that reflect precautionary saving.

We can generalize this intuitive line of reasoning.

2.2. General framework

We aim at characterizing the solution for jump variables $y_t \in \mathbb{R}^{n_y}$ and states $z_t \in \mathbb{R}^{n_z}$ of the dynamic system of equilibrium conditions with generic form:

$$\begin{aligned} 0 &= \ln E_t e^{f(y_t, z_t, y_{t+1}, z_{t+1})}, \quad f(y_t, z_t, y_{t+1}, z_{t+1}) \doteq h(y_t, z_t) + f_3 \, y_{t+1} + f_4 \, z_{t+1} \\ z_{t+1} &= g(y_t, z_t) + \lambda(z_t)(y_{t+1} - E_t y_{t+1}) + \sigma(z_t) \varepsilon_{t+1} \end{aligned} \tag{1}$$

where $\lambda(z_t)(y_{t+1} - E_t y_{t+1})$ describes heteroskedastic *endogenous* risk that depends on innovations in jump variables and $\sigma(z_t)\varepsilon_{t+1}$ is exogenous risk. Jump and state variables are expressed in deviations from the respective deterministic steady-state values. Operator $\ln E_t e^{[\cdot]}$ is applied elementwise to a vector-valued map, with E_t the expectations operator conditioned on the history up to time-t of state variables. Functions $f : \mathbb{R}^{2n_y + 2n_z} \to \mathbb{R}^{n_y}$, $h : \mathbb{R}^{n_y + n_z} \to \mathbb{R}^{n_y}$, $g : \mathbb{R}^{n_y + n_z} \to \mathbb{R}^{n_z}$, $\lambda : \mathbb{R}^{n_z} \to \mathbb{R}^{n_z \times n_y}$ and $\sigma : \mathbb{R}^{n_z} \to \mathbb{R}^{n_z \times n_\varepsilon}$ are differentiable. We denote by $f_i, g_i, ...$ the derivatives of $f, g, ...$ with respect to the ith argument; function f is linear in y_{t+1} and z_{t+1}. The equilibrium conditions of most DSGE models can be cast into this framework after suitable redefinition of variables.[9]

[9]The main loss of generality in this representation is the requirement that forward-looking arguments of the expectations operator be strictly positive, which is a key property to allow a connection with entropy.

Exogenous shocks $\varepsilon_t \in \mathbb{R}^{n_\varepsilon}$ form a martingale difference sequence with distribution described by the differentiable, conditional cumulant generating function (ccgf):[10]

$$\kappa[\alpha(z_t); z_t] \doteq \ln E_t e^{\alpha(z_t)'\varepsilon_{t+1}}, \quad \text{for any differentiable map } \alpha : \mathbb{R}^{n_z} \to \mathbb{R}^{n_\varepsilon}$$

2.3. *Affine approximation*

We are interested in approximating with an affine map the solution of system (1) for jump and state variables. Without loss of generality, we rewrite forward-looking equations as:

$$0 = h(y_t, z_t) + f_3 E_t y_{t+1} + f_4 E_t z_{t+1} + \mathcal{V}_t(e^{f_3 y_{t+1} + f_4 z_{t+1}}) \tag{2}$$

where $\mathcal{V}_t(e^{x_{t+1}}) \doteq \ln E_t e^{x_{t+1}} - E_t x_{t+1}$ is a relative entropy measure that breaks certainty equivalence.

We are looking for an affine solution $y_t = \widetilde{y} + \widetilde{\Psi}(z_t - \widetilde{z})$ with unknown coefficients $\widetilde{y}$, $\widetilde{\Psi}$ and $\widetilde{z}$. If $I_{n_z} - \lambda(z_t)\widetilde{\Psi}$ is a.s.-invertible, then innovations to the state vector have the approximate form:

$$z_{t+1} - E_t z_{t+1} = \lambda(z_t)\widetilde{\Psi}(z_{t+1} - E_t z_{t+1}) + \sigma(z_t)\varepsilon_{t+1} = (I_{n_z} - \lambda(z_t)\widetilde{\Psi})^{-1}\sigma(z_t)\varepsilon_{t+1}$$

In this context, rational expectations consistent with the affine solution imply the existence of a nonnegative function $\widetilde{\mathcal{V}} : \mathbb{R}^{n_z} \to \mathbb{R}^{n_y}_+$ of the state vector:

$$\widetilde{\mathcal{V}}(z_t) \doteq \mathcal{V}_t(e^{(f_3\widetilde{\Psi} + f_4) z_{t+1}}) = \kappa[(f_3\widetilde{\Psi} + f_4)(I_{n_z} - \lambda(z_t)\widetilde{\Psi})^{-1}\sigma(z_t); z_t] \tag{3}$$

where the connection with the ccgf follows from the properties of entropy.

Finally, we linearize equations (2) and (3) around the point $[y_t; z_t] = [\widetilde{y}; \widetilde{z}]$ as:

$$\begin{aligned} 0 &= h(\widetilde{y}, \widetilde{z}) + \widetilde{f}_1(y_t - \widetilde{y}) + \widetilde{f}_2(z_t - \widetilde{z}) + f_3 E_t y_{t+1} + f_4 E_t z_{t+1} + \widetilde{\mathcal{V}}(\widetilde{z}) + \widetilde{\mathcal{V}}_1(\widetilde{z})(z_t - \widetilde{z}) \\ E_t z_{t+1} &= g(\widetilde{y}, \widetilde{z}) + \widetilde{g}_1(y_t - \widetilde{y}) + \widetilde{g}_2(z_t - \widetilde{z}) \end{aligned}$$

with the notation $\widetilde{f}_i \doteq f_i(\widetilde{y}, \widetilde{z}, \widetilde{y}, \widetilde{z})$ and $\widetilde{g}_i \doteq g_i(\widetilde{y}, \widetilde{z})$.

We can now match coefficients to identify the conjectured affine solution:

$$\begin{aligned} y_t &= \widetilde{y} + \widetilde{\Psi}(z_t - \widetilde{z}) \\ z_{t+1} &= \widetilde{z} + \widetilde{g}_1(y_t - \widetilde{y}) + \widetilde{g}_2(z_t - \widetilde{z}) + (I_{n_z} - \lambda(z_t)\widetilde{\Psi})^{-1}\sigma(z_t)\varepsilon_{t+1} \end{aligned} \tag{4}$$

where the unknowns $[\widetilde{y}, \widetilde{z}, \widetilde{\Psi}]$ solve the system of equations:

$$\begin{aligned} 0 &= g(\widetilde{y}, \widetilde{z}) - \widetilde{z} \\ 0 &= h(\widetilde{y}, \widetilde{z}) + f_3\widetilde{y} + f_4\widetilde{z} + \widetilde{\mathcal{V}}(\widetilde{z}) \\ 0 &= \widetilde{f}_1\widetilde{\Psi} + \widetilde{f}_2 + (f_3\widetilde{\Psi} + f_4)(\widetilde{g}_1\widetilde{\Psi} + \widetilde{g}_2) + \widetilde{\mathcal{V}}_1(\widetilde{z}) \end{aligned} \tag{5}$$

[10] For example, if $\varepsilon_t \sim Niid(0, I)$, one has $\kappa[\alpha(z_t); z_t] = .5 diag[\alpha(z_t)\alpha(z_t)']$. In the special case of Gaussian shocks, a constant function λ and linear functions h and g, our affine approximation reduces to the one in Malkhozov (2014).

Crucially, the entropy terms represent risk corrections to an otherwise standard linearization.

Heuristically, general affine approximations can be summarized in the following algorithm:

Algorithm. *With system* (1) *as a starting point, proceed stepwise:*

Step 1. *Write expectations as the sum of a certainty-equivalent and an entropy terms as in* (2).

Step 2. *Conjecture a solution affine in the state vector and use it to characterize entropy as in* (3).

Step 3. *Identify the affine solution* (4) *by solving matrix equation* (5)*; or equivalently: i) linearize equations g and h around the point $[y_t; z_t] = [\widetilde{y}; \widetilde{z}]$; linearize the entropy term around the point $z_t = \widetilde{z}$ to derive an affine risk adjustment; ii) match coefficients to identify the unknowns $[\widetilde{y}, \widetilde{z}, \widetilde{\Psi}]$ of the conjectured solution.*

Note that, effectively, constant terms $[\widetilde{y}, \widetilde{z}]$ and dynamic coefficient $\widetilde{\Psi}$ are identified jointly only at the end of the algorithm. Expression (5) includes nonlinear matrix equations in the unknown coefficients that are amenable to straightforward Newton-type numerical solution methods.[11] However, these matrix equations are sufficiently nonlinear to allow for multiple solutions and to complicate the characterization of the local uniqueness of the constant terms and of the determinacy of the dynamics of the affine solution. Section 4 provides this characterization by drawing a link between affine approximations and perturbations around the risky steady state. In particular, constant terms $[\widetilde{y}, \widetilde{z}]$ acquire an appealing economic interpretation as a (saddle-point stable) risky steady state.

2.4. Incorporating terminal conditions

The representation of forward-looking *difference* equations in system (1) is ambiguous. When these equations are part of the equilibrium conditions they come in pairs with a terminal condition that allows for unwinding them forward into an infinite summation of time-t auxiliary variables (often interpreted as claims to single payoffs, so called strips). These variables obey forward-looking equations whose representation in system (1) is unambiguous. In contrast with conventional riskless perturbations, it matters whether we retain difference or summation specifications because they imply different assumptions about which variables are approximated as affine in the states, and that in turn affects entropy calculations.[12]

Formally there is no ambiguity: system (1) should include the summation specification because it embeds also the terminal condition. However, its richer structure may complicate the analysis when forward-looking difference equations affect the equilibrium allocation (e.g., as in production economies with capital accumulation), as the vector of equilibrium conditions would become infinite-dimensional. Therefore, in practice we advocate to first use a difference specification to determine the equilibrium allocation; then to use the result to price the auxiliary variables in a separate step; and finally to update the solution via the summation specification.

[11] The online appendix discusses a simple numerical algorithm to solve matrix equation (5).

[12] This remark applies also to perturbations around the risky steady state by the results of section 4.

3. Three didactic examples

We illustrate our affine approximation in the context of three simple models, and compare it with conventional perturbations. We start by pricing the consumption portfolio in the Gaussian endowment economy of Campbell and Cochrane (1999). We then show how to handle a non-Gaussian disaster component by pricing the risk-free bond in the model of Wachter (2013). Finally, we open borders and show how the solution to a savings problem in a small open economy with an exogenous risk-free bond associates with invalid deterministic steady state perturbations and yet a valid affine approximation.

In what follows lower-case letters and hat variables will denote respectively logarithms and log deviations from the expansion point.

3.1. Nonlinear habits in an endowment economy

A representative consumer with Campbell and Cochrane (1999) preferences:

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma}$$

lives in an endowment economy that describes the equilibrium risk-free rate (r) and the log wealth price-consumption ratio (pc) as a function of two exogenous state variables—the consumption process (C) and a process for surplus consumption ($S \doteq 1 - X/C$) relative to an external habit level (X). Parameter β is the rate of time preference and γ controls the curvature of the utility function.

This endowment economy is described by the pricing equation for the risk-free rate (priced in section 2.1) and by the pricing equation for the wealth portfolio:

$$e^{pc_t} = E_t e^{\ln(\beta)+(1-\gamma)\Delta c_{t+1}-\gamma\Delta s_{t+1}+\ln(1+e^{pc_{t+1}})} \tag{6}$$

When combined with a terminal condition that rules out bubbles, forward-looking difference equation (6) unwinds as:

$$pc_t = \ln\left(\sum_{n=1}^{\infty} e^{pc_t^{(n)}}\right), \quad pc_t^{(n)} = \ln E_t e^{\ln(\beta)+(1-\gamma)\Delta c_{t+1}-\gamma\Delta s_{t+1}+pc_{t+1}^{(n-1)}}, \quad pc_t^{(0)} = 0 \tag{7}$$

where $pc_t^{(n)}$ describes the log price-consumption ratio of the nth consumption strip, i.e., a claim to n-period ahead consumption. Including difference specification (6) among the equilibrium conditions implies approximating the price of the sum of strips as conditionally lognormal. Including summation specification (7) implies approximating each strip price as conditionally lognormal. Since the sum of lognormals is generically not a lognormal, and in contrast with conventional riskless perturbations, it follows that it matters which specification we choose to approximate.

3.1.1. Affine approximation

We use the algorithm in section 2 to obtain a linearization of the Euler equation and of equation (7), which embeds the terminal condition and hence more information than specification (6).

Step 1. Write expectational equations in terms of a certainty-equivalent and an entropy terms:

$$0 = \ln(\beta) + (1-\gamma)E_t\Delta c_{t+1} - \gamma E_t \Delta s_{t+1} + E_t pc_{t+1}^{(n-1)} - pc_t^{(n)} + \mathcal{V}_t\left(e^{(1-\gamma)\Delta c_{t+1} - \gamma\Delta s_{t+1} + pc_{t+1}^{(n-1)}}\right)$$

Step 2. Conjecture affine solutions $pc_t^{(n)} = \widetilde{pc}^{(n)} + \widetilde{\psi}^{(n)}\hat{s}_t$ and use the Gaussian ccgf to characterize the entropy terms as:

$$\mathcal{V}_t\left(e^{(1-\gamma)\Delta c_{t+1} - \gamma\Delta s_{t+1} + pc_{t+1}^{(n-1)}}\right) = \left(1 - \gamma[1 + \Lambda(\hat{s}_t)] + \widetilde{\psi}^{(n-1)}\Lambda(\hat{s}_t)\right)^2 \frac{\sigma^2}{2}$$

Step 3.i. Linearize:

$$\begin{aligned} 0 &= \ln(\beta) + (1-\gamma)E_t\Delta c_{t+1} - \gamma E_t \Delta s_{t+1} + E_t pc_{t+1}^{(n-1)} - pc_t^{(n)} + \left(1 - \gamma[1 + \Lambda(\hat{s}_t)] + \widetilde{\psi}^{(n)}\Lambda(\hat{s}_t)\right)^2 \frac{\sigma^2}{2} \\ &\approx \widetilde{pc}^{(n-1)} - \widetilde{pc}^{(n)} + \ln(\beta e^{(1-\gamma)\mu}) + \left(1 - \gamma[1 + \Lambda(0)] + \widetilde{\psi}^{(n-1)}\Lambda(0)\right)^2 \frac{\sigma^2}{2} \\ &\quad + \left[\widetilde{\psi}^{(n-1)}\phi - \widetilde{\psi}^{(n)} + \gamma(1-\phi) + \left(1 - \gamma[1 + \Lambda(0)] + \widetilde{\psi}^{(n-1)}\Lambda(0)\right)(\widetilde{\psi}^{(n-1)} - \gamma)\Lambda_1(0)\sigma^2\right]\hat{s}_t \end{aligned}$$

Step 3.ii. Match coefficients to identify the unknown vector $[\widetilde{r}; \widetilde{\psi}_r; \widetilde{pc}^{(n)}; \widetilde{\psi}^{(n)}]$ as:

$$\begin{aligned} \widetilde{pc}^{(n)} &= \widetilde{pc}^{(n-1)} + \ln(\beta e^{(1-\gamma)\mu}) + \left(1 - \gamma[1 + \Lambda(0)] + \widetilde{\psi}^{(n-1)}\Lambda(0)\right)^2 \frac{\sigma^2}{2} \\ \widetilde{\psi}^{(n)} &= \widetilde{\psi}^{(n-1)}\phi + \gamma(1-\phi) + \left(1 - \gamma[1 + \Lambda(0)] + \widetilde{\psi}^{(n-1)}\Lambda(0)\right)(\widetilde{\psi}^{(n-1)} - \gamma)\Lambda_1(0)\sigma^2 \end{aligned}$$

with boundary condition $\widetilde{pc}^{(0)} = \widetilde{\psi}^{(0)} = 0$. The approximate solution is:

$$\hat{s}_{t+1} = \phi\hat{s}_t + \Lambda(\hat{s}_t)\sigma\varepsilon_{t+1} \tag{8}$$

$$pc_t = \ln\left(\sum_{n=1}^{\infty} e^{pc_t^{(n)}}\right), \quad pc_t^{(n)} = \widetilde{pc}^{(n)} + \widetilde{\psi}^{(n)}\hat{s}_t \tag{9}$$

3.1.2. Perturbations around the deterministic steady state

It is instructive to compare our approximation with a conventional third-order approximation:

$$\hat{s}_{t+1} = \phi\hat{s}_t + \Lambda(0)\sigma\varepsilon_{t+1} + \underbrace{\Lambda_1(0)\hat{s}_t\sigma\varepsilon_{t+1}}_{\text{2nd order term}} + \underbrace{\frac{1}{2}\Lambda_{11}(0)\hat{s}_t^2\sigma\varepsilon_{t+1}}_{\text{3rd order term}} \tag{10}$$

$$r_t = -\ln(\beta e^{-\gamma\mu}) - \gamma(1-\phi)\hat{s}_t - \underbrace{\gamma^2[1 + \Lambda(0)]^2\frac{\sigma^2}{2}}_{\text{2nd order term}} - \underbrace{\gamma^2[1 + \Lambda(0)]\Lambda_1(0)\sigma^2\hat{s}_t}_{\text{3rd order term}}$$

$$pc_t = \ln\left(\sum_{n=1}^{\infty} e^{pc_t^{(n)}}\right), \quad pc_t^{(n)} = n\ln(\beta e^{(1-\gamma)\mu}) + \bar{\psi}_1^{(n)}\hat{s}_t + \underbrace{\bar{\psi}_2^{(n)}}_{\text{2nd order term}} + \underbrace{\bar{\psi}_3^{(n)}\hat{s}_t}_{\text{3rd order term}} \tag{11}$$

Parameter	value
Subjective discount factor, β	$.89^{1/12}$
Utility curvature, γ	2
Habit persistence, ϕ	$.87^{1/12}$
Mean growth rate (in %), μ	$1.89/12$
Standard deviation of consumption innovations (in %), σ_c	$1.50/\sqrt{12}$
Standard deviation of dividend innovations (in %), σ_d	$11.2/\sqrt{12}$
Consumption-dividend correlation, ρ	.2

Table 1: Deep parameters and their calibration (monthly frequency) in Campbell and Cochrane (1999).

where $\bar{\psi}_1^{(n)} = \gamma(1-\phi^n)$ and:

$$\bar{\psi}_2^{(n)} = \bar{\psi}_2^{(n-1)} + \left(1 - \gamma[1+\Lambda(0)] + \bar{\psi}_1^{(n-1)}\Lambda(0)\right)^2 \frac{\sigma^2}{2} \qquad \bar{\psi}_2^{(0)} = 0$$

$$\bar{\psi}_3^{(n)} = \bar{\psi}_3^{(n-1)}\phi + \left(1 - \gamma[1+\Lambda(0)] + \bar{\psi}_1^{(n-1)}\Lambda(0)\right)(\bar{\psi}_1^{(n-1)} - \gamma)\Lambda_1(0)\sigma^2, \qquad \bar{\psi}_3^{(0)} = 0$$

A second-order perturbation captures a constant precautionary savings motive. However, it misses the dynamics of the component, which show up only to third order. In this simple example, the equilibrium dependence of decision variables on states under a conventional third-order perturbation and under an affine approximation coincide for the risk-free rate but they differ already for the equilibrium price of consumption strips. Proposition 2 formalizes the extent of this distinction between affine approximations and conventional perturbations.

Note that the representation of innovations in state variables is also different by comparing (8) and (10); the affine method does not approximate $\Lambda(s_t)$ to represent surplus-consumption news. In fact, the small radius of convergence of function $\sqrt{1-2\hat{s}_t}$ when approximated around $\hat{s}_t = 0$ implies that higher-order approximations are especially inaccurate when simulating tail regions of the state space associated with values $s_t < .5$ that matter most for pricing.

3.1.3. Numerical example

We specify sensitivity function $\Lambda(\hat{s}_t) = \sqrt{\frac{(1-\phi)(1-2\hat{s}_t)}{\gamma var(c_t - E_{t-1}c_t)}} - 1$ and calibrate the model using the values in Campbell and Cochrane (1999) reported in table 1. Figure 1 compares the exact solution to our affine approximation by plotting the map from the value of the state variables (surplus consumption) into the price-dividend ratio of consumption portfolios. We consider the wealth portfolio (a claim to the future consumption stream) and the market portfolio (a claim to the future dividend stream). We consider two local approximations of price-dividend ratios. We consider the map (9), which retains some nonlinearities in surplus consumption; and the affine approximation of difference specification (6), which does not incorporate information from the terminal condition.

The maps from the state space into the portfolio price-dividend ratios are very similar in shape except for the level when comparing global and affine methods. Wachter (2006) warned us of a difference in level apparent when using an insufficiently coarse grid; we see the same effect in figure 1. In this sense, our generalized affine method is similar to a global solution with a relatively small number of grid points. We emphasize the importance of pricing portfolios from the price

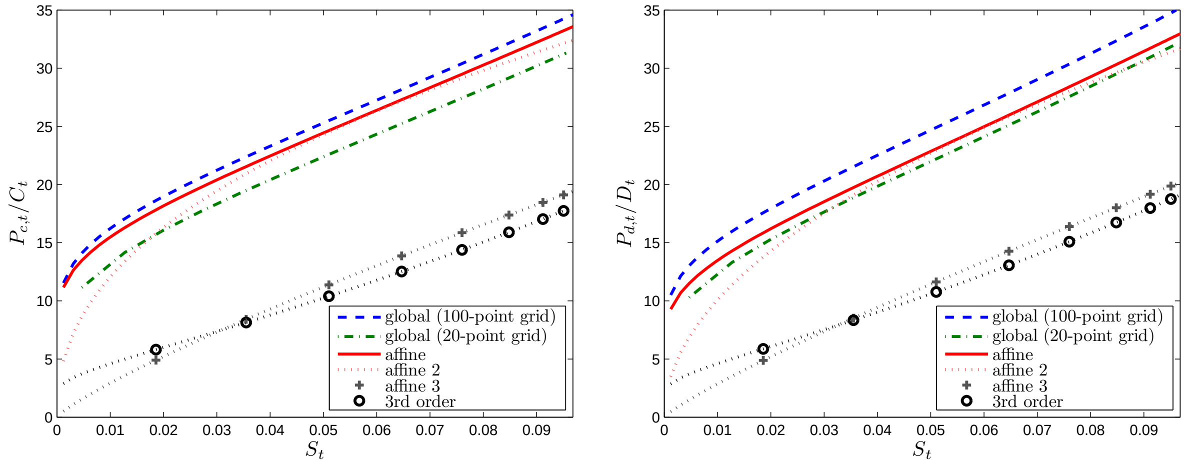

Figure 1: Comparison of solutions for the map of surplus consumption into the price-dividend ratio of consumption (P_c/C) and market equity (P_d/D) portfolios in the model of Campbell and Cochrane (1999). Projected solution, affine approximation and third-order perturbation around the deterministic steady state. 'Affine' and '3rd order' work with expressions (9) and (11); 'affine 2' linearizes expression (9) around $\hat{s}_t = 0$. 'Affine 3' approximates difference specification (6). Global solutions use cubic splines collocated over 100 and 20 Chebyshev nodes and 20- and 5-point Gauss-Hermite quadrature, respectively.

of strips using maps (7) and (9). In contrast, conventional third-order perturbations as well as the alternative local method based on difference specification (6) deliver a much less accurate approximation as they capture to a lower extent the nonlinear effect of risk on prices.

Overall, perhaps the most remarkable feature of figure 1 is that affine approximations offer an accurate solution; Campbell-Cochrane habits present a less nonlinear problem than we thought.

3.2. Disaster risk in an endowment economy

We consider a discrete-time version of the endowment economy in Wachter (2013). Investors have Epstein-Zin-Weil recursive preferences:

$$v_t = c_t + \frac{1}{1-\rho} \ln\left(1 - \beta + \beta e^{(1-\rho)(x_t - c_t)}\right), \quad x_t \doteq \frac{1}{1-\gamma} \ln E_t e^{(1-\gamma)v_{t+1}}$$

with elasticity of intertemporal substitution ρ and risk aversion coefficient γ. They live in an endowment economy in which log consumption growth has a normal component ε^c as well as a disaster component ξ modeled as a Poisson mixture of normals:

$$c_{t+1} = \mu + c_t + \sigma\varepsilon^c_{t+1} - \theta\xi_{t+1}$$

with $\varepsilon^c_t \sim Niid(0, 1)$ and $\xi_t | j_t \sim N(j_t, j_t\delta^2)$, with the number of jumps $j_{t+1} \sim Poisson(p_t)$, hence $E_t\xi_{t+1} = E_t j_{t+1} = p_t$. We assume that ε^c_t and $\varepsilon^\xi_t \doteq \xi_t - E_{t-1}\xi_t$ are independent. Jump intensity p_t evolves according to the recursive law of motion:

$$p_{t+1} = (1-\rho_p)p + \rho_p p_t + \sqrt{p_t}\phi_p\sigma\varepsilon^p_{t+1}$$

with $\varepsilon_t^p \sim Niid(0,1)$ and independent of ε_t^c and ε_t^ξ. It follows that shock $\varepsilon_t = [\varepsilon_t^c; \varepsilon_t^p; \varepsilon_t^\xi]$ has ccgf:

$$\kappa([\alpha_c;\alpha_p;\alpha_\xi];p_t) = \frac{1}{2}\alpha_c^2 + \frac{1}{2}\alpha_p^2 + \left[(e^{\alpha_\xi + \alpha_\xi^2\frac{\delta^2}{2}} - 1) - \alpha_\xi\right] p_t$$

The economy is described by the definition of preferences and by the Euler equation describing the equilibrium risk-free rate:

$$\begin{aligned}
0 &= \frac{1}{1-\rho}\ln\left(1-\beta+\beta e^{(1-\rho)xc_t}\right) - vc_t \\
0 &= \frac{1}{1-\gamma}\ln E_t e^{(1-\gamma)(vc_{t+1}+\Delta c_{t+1})} - xc_t \\
0 &= \ln E_t e^{\ln(\beta)-\gamma\Delta c_{t+1}+(\rho-\gamma)(vc_{t+1}-xc_t)+r_t}
\end{aligned}$$

where $vc_t \doteq v_t - c_t$ and $xc_t \doteq x_t - c_t$ are detrended variables.

3.2.1. Affine approximation

Step 1. Write expectational equations in terms of a certainty-equivalent and an entropy terms:

$$\begin{aligned}
0 &= E_t vc_{t+1} + \mu - \theta p_t + \frac{1}{1-\gamma}\mathcal{V}_t\left(e^{(1-\gamma)\left(vc_{t+1}+\sigma\varepsilon_{t+1}^c-\theta\varepsilon_{t+1}^\xi\right)}\right) - xc_t \\
0 &= \ln(\beta e^{-\gamma\mu}) + \gamma\theta p_t + (\rho-\gamma)(E_t vc_{t+1} - xc_t) + \mathcal{V}_t\left(e^{-\gamma\sigma\varepsilon_{t+1}^c+\gamma\theta\varepsilon_{t+1}^\xi+(\rho-\gamma)vc_{t+1}}\right) + r_t
\end{aligned}$$

Step 2. Conjecture an affine solution $r_t = \widetilde{r} + \widetilde{\psi}_r\hat{p}_t$, $vc_t = \widetilde{vc} + \widetilde{\psi}_v\hat{p}_t$ and $xc_t = \widetilde{xc} + \widetilde{\psi}_x\hat{p}_t$ and combine it with the ccgf to characterize the entropy terms:

$$\begin{aligned}
\mathcal{V}_t\left(e^{(1-\gamma)\left(vc_{t+1}+\sigma\varepsilon_{t+1}^c-\theta\varepsilon_{t+1}^\xi\right)}\right) &= \frac{(1-\gamma)^2\sigma^2}{2} + \left[\frac{(1-\gamma)^2\widetilde{\psi}_v^2\phi_p^2\sigma^2}{2} + e^{(\gamma-1)\theta+\frac{(\gamma-1)^2\theta^2\delta^2}{2}} - 1 + (1-\gamma)\theta\right] p_t \\
\mathcal{V}_t\left(e^{-\gamma\sigma\varepsilon_{t+1}^c+\gamma\theta\varepsilon_{t+1}^\xi+(\rho-\gamma)vc_{t+1}}\right) &= \frac{\gamma^2\sigma^2}{2} + \left[\frac{(\rho-\gamma)^2\widetilde{\psi}_v^2\phi_p^2\sigma^2}{2} + e^{\gamma\theta+\frac{\gamma^2\theta^2\delta^2}{2}} - 1 - \gamma\theta\right] p_t
\end{aligned}$$

Step 3.i. Linearize:

$$\begin{aligned}
0 &= \frac{1}{1-\rho}\ln\left(1-\beta+\beta e^{(1-\rho)\widetilde{xc}}\right) - \widetilde{vc} + \left(\frac{\beta e^{(1-\rho)\widetilde{xc}}}{1-\beta+\beta e^{(1-\rho)\widetilde{xc}}}\widetilde{\psi}_x - \widetilde{\psi}_v\right)\hat{p}_t \\
0 &= \widetilde{vc} + \mu - \widetilde{xc} + (\widetilde{\psi}_v\rho_p - \widetilde{\psi}_x)\hat{p}_t + \frac{(1-\gamma)\sigma^2}{2} + \left[\frac{(1-\gamma)\widetilde{\psi}_v^2\phi_p^2\sigma^2}{2} + \frac{e^{(\gamma-1)\theta+\frac{(\gamma-1)^2\theta^2\delta^2}{2}} - 1}{1-\gamma}\right] p_t \\
0 &= \ln(\beta e^{-\gamma\mu}) + (\rho-\gamma)(\widetilde{vc} - \widetilde{xc}) + \widetilde{r} + \left[(\rho-\gamma)(\widetilde{\psi}_v\rho_p - \widetilde{\psi}_x) + \widetilde{\psi}_r\right]\hat{p}_t + \frac{\gamma^2\sigma^2}{2} + \left[\frac{(\rho-\gamma)^2\widetilde{\psi}_v^2\phi_p^2\sigma^2}{2} + e^{\gamma\theta+\frac{\gamma^2\theta^2\delta^2}{2}} - 1\right] p_t
\end{aligned}$$

Step 3.ii. Match coefficients to identify the unknown vector $[\widetilde{r}; \widetilde{\psi}_r; \widetilde{vc}; \widetilde{\psi}_v; \widetilde{xc}; \widetilde{\psi}_x]$:

$$\widetilde{vc} = -\frac{1}{1-\rho}\ln\left[\frac{1-\beta e^{(1-\rho)(\mu-\nu_{0,1})}}{1-\beta}\right] \qquad \widetilde{\psi}_v = \frac{1-\sqrt{1-2\left(\frac{\beta e^{(1-\rho)\widetilde{x}}\phi_p\sigma}{1-\beta+\beta e^{(1-\rho)\widetilde{x}}(1-\rho_p)}\right)^2\left(e^{(\gamma-1)\theta+\frac{(\gamma-1)^2\theta^2\delta^2}{2}}-1\right)}}{\frac{\beta e^{(1-\rho)\widetilde{x}}(\gamma-1)}{1-\beta+\beta e^{(1-\rho)\widetilde{x}}(1-\rho_p)}\phi_p^2\sigma^2}$$

$$\widetilde{xc} = \frac{1}{1-\rho}\ln\left[\frac{(1-\beta)e^{(1-\rho)(\mu-\nu_{0,1})}}{1-\beta e^{(1-\rho)(\mu-\nu_{0,1})}}\right] \qquad \nu_{0,1} \doteq \frac{(\gamma-1)\sigma^2}{2} + \frac{(\gamma-1)\widetilde{\psi}_v^2\phi_p^2\sigma^2}{2}p + \frac{e^{(\gamma-1)\theta+\frac{(\gamma-1)^2\theta^2\delta^2}{2}}-1}{\gamma-1}p$$

$$\widetilde{\psi}_x = \frac{1-\beta+\beta e^{(1-\rho)\widetilde{x}}}{\beta e^{(1-\rho)\widetilde{x}}}\widetilde{\psi}_v \qquad \nu_{0,2} \doteq \frac{\gamma^2\sigma^2}{2} + \frac{(\gamma-\rho)^2\widetilde{\psi}_v^2\phi_p^2\sigma^2}{2}p + \left(e^{\gamma\theta+\frac{\gamma^2\theta^2\delta^2}{2}}-1\right)p$$

$$\widetilde{r} = -\ln(\beta e^{-\rho\mu}) + (\gamma-\rho)\nu_{0,1} - \nu_{0,2}, \qquad \nu_{p,1} \doteq \frac{(\gamma-1)\widetilde{\psi}_v^2\phi_p^2\sigma^2}{2} + \frac{e^{(\gamma-1)\theta+\frac{(\gamma-1)^2\theta^2\delta^2}{2}}-1}{\gamma-1}$$

$$\widetilde{\psi}_r = (\gamma-\rho)\nu_{p,1} - \nu_{p,2} \qquad \nu_{p,2} \doteq \frac{(\gamma-\rho)^2\widetilde{\psi}_v^2\phi_p^2\sigma^2}{2} + e^{\gamma\theta+\frac{\gamma^2\theta^2\delta^2}{2}} - 1$$

where only the solution associated with the negative root implies $\widetilde{\psi}_v = 0$ if disasters have no impact.

The affine method identifies separately three components of the risk-free rate: an intertemporal substitution motive captured by the term $-\ln(\beta e^{-\rho\mu})$, a precautionary savings motive captured by the term $\nu_{0,2} + \nu_{p,2}\hat{p}_t$, and a resolution of uncertainty motive captured by the term $(\gamma-\rho)(\nu_{0,1} + \nu_{p,1}\hat{p}_t)$.

In this context, we limit ourselves to comparing our approximation with the exact solution in the limit as $\rho \to 1$, in which the following closed-form solution is available:

$$vc_t = \frac{\beta(\mu-\nu_{0,1})}{1-\beta} + \widetilde{\psi}_v\hat{p}_t \qquad \widetilde{\psi}_v = \frac{1-\sqrt{1-2\left(\frac{\beta\phi_p\sigma}{1-\beta\rho_p}\right)^2\left(e^{(\gamma-1)\theta+\frac{(\gamma-1)^2\theta^2\delta^2}{2}}-1\right)}}{\frac{\beta(\gamma-1)}{1-\beta\rho_p}\phi_p^2\sigma^2}$$

$$xc_t = \frac{\mu-\nu_{0,1}}{1-\beta} + \frac{1}{\beta}\widetilde{\psi}_v\hat{p}_t \qquad \nu_{0,1} \doteq \frac{(\gamma-1)\sigma^2}{2} + \frac{(\gamma-1)\widetilde{\psi}_v^2\phi_p^2\sigma^2}{2}p + \frac{e^{(\gamma-1)\theta+\frac{(\gamma-1)^2\theta^2\delta^2}{2}}-1}{\gamma-1}p$$

$$r_t = -\ln(\beta e^{-\mu}) + \frac{(1-2\gamma)\sigma^2}{2} + \widetilde{\psi}_r(p+\hat{p}_t), \qquad \widetilde{\psi}_r = e^{(\gamma-1)\theta+\frac{(\gamma-1)^2\theta^2\delta^2}{2}} - e^{\gamma\theta+\frac{\gamma^2\theta^2\delta^2}{2}}$$

The affine approximation recovers the exact solution. Similarly, when $\rho \to 1$ our method prices exactly the term structures of consumption equity and real interest rates.

3.2.2. Perturbations around the deterministic steady state

While our approximation recovers the exact solution for all affine distributions, perturbations do so only asymptotically. Even in the simple case with time-separable preferences ($\gamma = \rho$), as $\rho \to 1$ a conventional Nth-order perturbation of the system yields:

$$r_t = -\ln(\beta e^{-\mu}) - \theta p_t - \sum_{j=1}^{N}\frac{\kappa_{j,t}}{j!} \quad \overset{N\to\infty}{\to} \quad -\ln(\beta e^{-\mu}) - \frac{\sigma^2}{2} - \left(e^{\theta+\theta^2\frac{\delta^2}{2}}-1\right)p_t$$

where $\kappa_{j,t}$ is the jth conditional cumulant of $-(\sigma\varepsilon^c_{t+1} - \theta\varepsilon^\xi_{t+1})$, hence $\ln E_t e^{-(\sigma\varepsilon^c_{t+1}-\theta\varepsilon^\xi_{t+1})} = \sum_{j=1}^{\infty}\frac{\kappa_{j,t}}{j!}$.

3.3. Portfolio choice in a small open economy

The next example turns off habits and opens borders. We borrow the example from Coeurdacier et al. (2011) as a case with an invalid local approximation around the deterministic steady state but a well-defined risky steady state. An affine approximation is also well-defined; in fact, section 4 shows the equivalence between our approximation and a risky steady state perturbation.

The representative consumer lives in a small open economy, receives an exogenous stream of income $Y_t = \theta$ for $t = 0, 1, ...,$ and chooses a nonnegative consumption stream by investing an amount A_t in foreign assets that pay off an exogenous interest rate r with law of motion:

$$r_{t+1} - r = \rho(r_t - r) + \sigma_r \varepsilon^r_{t+1}, \quad \varepsilon^r_t \sim Niid(0, 1)$$

Investment is subject to a no-Ponzi condition. Markets are incomplete in that no other financial claim exists; the budget constraint is the backward-looking difference equation:

$$C_t + A_t \leq Y_t + e^{r_t} A_{t-1}, \quad \lim_{h\to\infty} \left(\Pi^h_{s=1} e^{-r_{t+s}}\right) A_{t+h} \geq 0$$

As required for a finite, stationary solution for consumption and asset holdings we assume $\beta e^r < 1$ (see Chamberlain and Wilson, 2000).

Joint optimality of consumption and foreign-asset holdings implies:

$$\begin{aligned} 0 &\geq \ln E_t e^{\ln(\beta) - \gamma\Delta c_{t+1} + r_{t+1}}, \quad = \text{if } A_t > 0 \\ A_t &= e^{r_t} A_{t-1} + \theta - C_t \end{aligned}$$

3.3.1. Affine approximation

We define variables $B_t \doteq \exp(A_t)$, $X^c_t \doteq E_t C_{t+1}$, $X^r_t \doteq E_t R_{t+1}$, $W^c_t \doteq \exp(C_t)$ and $W^r_t \doteq \exp(R_t)$ to recast the problem in form (1) as:

$$0 = \ln E_t e^{f(y_t, z_t, y_{t+1}, z_{t+1})}, \quad f(y_t, z_t, y_{t+1}, z_{t+1}) = \begin{bmatrix} \ln(\beta) - \gamma\Delta c_{t+1} + r_{t+1} \\ c_{t+1} - x^c_t \\ r_{t+1} - x^r_t \\ e^{c_t} - w^c_t \\ e^{r_t} - w^r_t \end{bmatrix}$$

$$\begin{bmatrix} b_{t+1} \\ r_{t+1} \end{bmatrix} = \begin{bmatrix} \theta - e^{x^c_t} + b_t e^{x^r_t} \\ (1-\rho) r + \rho r_t \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -1 & b_t \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} (E_{t+1} - E_t) y_{t+1} + \begin{bmatrix} 0 \\ \sigma_r \end{bmatrix} \varepsilon^r_{t+1}$$

with decision variables $y_t = [c_t; x^c_t; x^r_t; w^c_t; w^r_t]$, states $z_t = [b_t; r_t]$ and shock $\varepsilon_t = \varepsilon^r_t \sim Niid(0, 1)$.

Step 1. Write expectational equations in terms of a certainty-equivalent and an entropy terms:

$$\begin{aligned} 0 &= \ln(\beta) - \gamma E_t \Delta c_{t+1} + E_t r_{t+1} + \mathcal{V}_t \left(e^{-\gamma\Delta c_{t+1} + r_{t+1}}\right) \\ 0 &= E_t c_{t+1} - x^c_t + \mathcal{V}_t \left(e^{c_{t+1}}\right) \\ 0 &= E_t r_{t+1} - x^r_t + \mathcal{V}_t \left(e^{r_{t+1}}\right) \end{aligned}$$

Step 2. Conjecture a linear solution $y_t = \widetilde{y} + \widetilde{\psi}_{yz}(z_t - \widetilde{z})$, hence:

$$\mathcal{V}_t\left(e^{-\gamma\Delta c_{t+1}+r_{t+1}}\right) = \left(1-\gamma\frac{e^r\widetilde{\psi}_{cb}b_t+\widetilde{\psi}_{cr}}{1+e^{\widetilde{c}}\widetilde{\psi}_{cb}}\right)^2\frac{\sigma_r^2}{2},\quad \mathcal{V}_t\left(e^{c_{t+1}}\right) = \left(\frac{e^r\widetilde{\psi}_{cb}b_t+\widetilde{\psi}_{cr}}{1+e^{\widetilde{c}}\widetilde{\psi}_{cb}}\right)^2\frac{\sigma_r^2}{2},\quad \mathcal{V}_t\left(e^{r_{t+1}}\right) = \frac{\sigma_r^2}{2}$$

Step 3. Linearize and match coefficients or, equivalently, solve matrix equation (5) with:

$$\widetilde{h} = \begin{bmatrix}\ln(\beta)+\gamma\widetilde{c}\\ \widetilde{c}-\widetilde{x}^c\\ -\widetilde{x}^r\\ e^{\widetilde{c}}-\widetilde{w}^c\\ e^r-\widetilde{w}^r\end{bmatrix},\quad \widetilde{g} = \begin{bmatrix}\theta - e^{\widetilde{x}^c}+\widetilde{b}e^r\\ r\end{bmatrix},\quad \widetilde{\mathcal{V}}(\widetilde{z}) = \begin{bmatrix}\left(1-\gamma\frac{e^r\widetilde{\psi}_{cb}\widetilde{b}+\widetilde{\psi}_{cr}}{1+e^{\widetilde{c}}\widetilde{\psi}_{cb}}\right)^2\frac{\sigma_r^2}{2}\\ \left(\frac{e^r\widetilde{\psi}_{cb}\widetilde{b}+\widetilde{\psi}_{cr}}{1+e^{\widetilde{c}}\widetilde{\psi}_{cb}}\right)^2\frac{\sigma_r^2}{2}\\ \frac{\sigma_r^2}{2}\\ 0\\ 0\end{bmatrix}$$

$$\widetilde{f}_1 = \begin{bmatrix}\gamma & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0\\ e^{\widetilde{c}} & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & -1\end{bmatrix},\quad \widetilde{f}_2 = \begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & e^r\end{bmatrix},\quad f_3 = \begin{bmatrix}-\gamma & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix},\quad f_4 = \begin{bmatrix}0 & 1\\ 0 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\end{bmatrix}$$

$$\widetilde{g}_1 = \begin{bmatrix}0 & -e^{\widetilde{x}^c} & \widetilde{b}e^{\widetilde{x}^r} & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix},\quad \widetilde{g}_2 = \begin{bmatrix}e^{\widetilde{x}^r} & 0\\ 0 & \rho\end{bmatrix},\quad \widetilde{\mathcal{V}}_1(\widetilde{z}) = \begin{bmatrix}\left(\gamma\frac{e^r\widetilde{\psi}_{cb}\widetilde{b}+\widetilde{\psi}_{cr}}{1+e^{\widetilde{c}}\widetilde{\psi}_{cb}}-1\right)\frac{\gamma e^r\widetilde{\psi}_{cb}}{1+e^{\widetilde{c}}\widetilde{\psi}_{cb}}\sigma_r^2 & 0\\ \frac{e^r\widetilde{\psi}_{cb}\widetilde{b}+\widetilde{\psi}_{cr}}{1+e^{\widetilde{c}}\widetilde{\psi}_{cb}}\frac{e^r\widetilde{\psi}_{cb}}{1+e^{\widetilde{c}}\widetilde{\psi}_{cb}}\sigma_r^2 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$$

One can show that the affine approximation results in saddle-path stable dynamics for a wide range of values of the deep parameters. Coeurdacier et al. (2011) discuss the stable dynamics of the model around the risky steady state, so our stability result is not surprising once we realize the equivalence of affine approximations and first-order perturbations around the risky steady state. We leave to section 4 the proof of this proposition and hence the formal justification of affine methods as well as the characterization of saddle-path conditions.

3.3.2. Perturbations around the deterministic steady state

Conventional perturbations around the deterministic steady state applied to this are not well-defined. In fact, in this example the deterministic steady state is the corner solution $C = 0$ and $A = -\frac{\theta}{e^r-1}$, and perturbations around that point that associate with $C < 0$ would be inadmissible.

We also rule out an interior solution, as it would imply a deterministic steady state with $\beta e^r = 1$, a contradiction. If we had assumed $\beta e^r = 1$, then the pair (c, a) would be indeterminate—any $A > -\frac{\theta}{e^r-1}$ would be a stationary value. Accordingly, we would have unit-root local dynamics that invalidate the local approximation (Schmitt-Grohé and Uribe, 2003).

4. Approximation method: Formal statement

4.1. Formal derivation

Proposition 1 provides the mathematical foundation for affine approximations. Part a. characterizes saddle-path stability of any solution to the nonlinear equations (5), adapting Blanchard and Kahn (1980) conditions to our context. Relative to conventional linearizations, the determinacy of equilibrium dynamics is affected by the evaluation of derivatives at $[y_t, z_t] = [\widetilde{y}, \widetilde{z}]$ and by the presence of the dynamic entropy component. Part b. shows how affine approximations coincide with linear perturbations around the risky steady state, thereby inheriting their formal justification based on the implicit function and Taylor theorems.

To set the ground for perturbations, for any $q \in [0, 1]$, let the point $[\widetilde{y}^q, \widetilde{z}^q]$, matrix Ψ^q and the differentiable function $\widetilde{\mathcal{V}}(z^q, q) \doteq \kappa[(f_3\Psi^q + f_4)(I - \lambda(z^q)\Psi^q)^{-1}\sigma(z^q)q; z^q]$ such that:

$$
\begin{aligned}
0 &= h(\widetilde{y}^q, \widetilde{z}^q) + f_3\widetilde{y}^q + f_4\widetilde{z}^q + \widetilde{\mathcal{V}}(\widetilde{z}^q, q), \quad \widetilde{z}^q = g(\widetilde{y}^q, \widetilde{z}^q) \\
0 &= f_1^q\Psi^q + f_2^q + (f_3\Psi^q + f_4)(g_1^q\Psi^q + g_2^q) + \widetilde{\mathcal{V}}_1(\widetilde{z}^q, q)
\end{aligned} \tag{12}
$$

where $f_i^q \doteq h_i(\widetilde{y}^q, \widetilde{z}^q)$ and $g_i^q \doteq g_i(\widetilde{y}^q, \widetilde{z}^q)$. At $q = 1$, we recover (5) and $\widetilde{y}^q = \widetilde{y}$, $\widetilde{z}^q = \widetilde{z}$ and $\Psi^q = \widetilde{\Psi}$. Consider the parametrized family of system (1):

$$
\begin{aligned}
0 &= E_t x_{t+1}^q + \tau\mathcal{V}_t[\exp(x_{t+1}^q)] + (1-\tau)\widetilde{\mathcal{V}}(z_t^q, q) \\
z_{t+1}^q &= g[y(z_t^q, q, \tau), z_t^q] + \lambda(z_t^q)(E_{t+1} - E_t)y[z(z_t^q, q, \varepsilon_{t+1}, \tau), q, \tau] + \sigma(z_t^q)q\varepsilon_{t+1} \\
x_{t+1}^q &\doteq h[y(z_t^q, q), z_t^q, \tau] + f_3 y[z(z_t^q, q, \varepsilon_{t+1}, \tau), q, \tau] + f_4 z(z_t^q, q, \varepsilon_{t+1}, \tau)
\end{aligned} \tag{13}
$$

We are looking for solutions for jump and state variables $y_t^q = y(z_t^q, q, \tau)$ and $z_{t+1}^q = z(z_t^q, q, \varepsilon_{t+1}, \tau)$, where the perturbation scalars q and τ index respectively the amount of risk in the economy and whether entropy is evaluated using the true or a linear function $y_t^q = \widetilde{y}^q + \Psi^q(z_t^q - \widetilde{z}^q)$. Under $q = \tau = 1$ the dynamics coincide with the original model (1). Entropy $w(z_t^q, q) \doteq \mathcal{V}(e^{x_{t+1}^q}|z_t^q)$ is assumed to be differentiable in z_t^q for all $q \in [0, 1]$.

Definition. A risky steady state of system (13) is a point $[\bar{y}^q; \bar{z}^q]$ such that $\bar{z}^q = z(\bar{z}^q, q, 0, 0)$ and $\bar{y}^q = y(\bar{z}^q, q, 0)$, i.e., a point where agents choose to stay *i)* while expecting shocks ex ante; *ii)* when ex-post realized shocks are zero; *iii)* while forming expectations consistent with a linear solution.

Proposition 1. (a) *A point* $(y_t^q, z_t^q) = (\widetilde{y}^q, \widetilde{z}^q)$ *that solves system* (12) *(and* (5) *at* $q = 1$*) associates with a saddle-path stable solution if and only if square matrices:*

$$
\Gamma \doteq \begin{bmatrix} f_4 & f_3 \\ I_{n_z} & 0 \end{bmatrix} \in \mathbb{C}^{n_y+n_z \times n_y+n_z}, \qquad \Xi^q \doteq \begin{bmatrix} -f_2^q - \widetilde{\mathcal{V}}_1(\widetilde{z}^q, q) & -f_1^q \\ g_2^q & g_1^q \end{bmatrix} \in \mathbb{C}^{n_y+n_z \times n_y+n_z}
$$

have n_z *generalized eigenvalues* $\alpha(\Gamma, \Xi^q) \doteq \{\alpha \in \mathbb{C} : \det(\Gamma\alpha - \Xi^q) = 0\}$ *inside the unit circle and* n_y *larger than unity. At* $q = 1$, $f_i^q = \widetilde{f}_i$, $g_i^q = \widetilde{g}_i$ *and* $\widetilde{\mathcal{V}}_1(\widetilde{z}^q, q) = \widetilde{\mathcal{V}}_1(\widetilde{z})$ *as defined in* (3).

(b) *Suppose that a risky steady state* $z_t^q = \bar{z}^q$ *is a saddle point of system* (13). *Then,* i) *functions* $y_t^q = y(z_t^q, q, \tau)$ *and* $z_{t+1}^q = z(z_t^q, q, \varepsilon_{t+1}, \tau)$ *are unique and differentiable in a neighborhood of the risky steady state; and* ii) *the affine approximate solution coefficients* $[\widetilde{y}^q, \widetilde{z}^q, \widetilde{\Psi}^q]$ *to system* (12) *coincide with the coefficients from a linear perturbation around the risky steady state:* $\widetilde{y}^q = \bar{y}^q$*,* $\widetilde{z}^q = \bar{z}^q$*,* $\Psi^q = y_1(\bar{z}^q, q, 0)$*. At* $q = 1$*, we have* $\widetilde{y} = \bar{y}^1$*,* $\widetilde{z} = \bar{z}^1$*,* $\widetilde{\Psi} = y_1(\bar{z}^1, 1, 0)$*.*

Appendices A and B provide a proof of the proposition.

An important difference between our method and perturbations remains in the characterization of innovations in state variables, described in (4), as we do not approximate the maps λ and σ defined in system (1). We find this strategy compelling when simulating the solution of DSGE models because the maps λ and σ in leading examples have Taylor series with a small radius of convergence. Approximations of those maps can result in spurious dynamics, with an inaccurate representation of tail regions of the state space that matter most for pricing, as low-probability regions under the physical probability can acquire a much larger probability mass under the risk-neutral measure.

4.2. Relationship with conventional perturbations

Our affine approximation (and hence a first-order perturbation around the risky steady state, by proposition 1) is not nested in conventional perturbations around the deterministic steady state (e.g., à la Schmitt-Grohé and Uribe, 2004).[13] Proposition 2 provides sufficient conditions under which nesting does not occur and takes stock of the lessons from the examples of section 3.

Note that one can at most reconstruct the implicit functions $y(0, q)$ and $y_1(0, q)$ using output from Nth-order perturbations around the deterministic steady state $(z_t, q) = (0, 0)$ as:

$$y(0,q) = \lim_{N\to\infty} \sum_{i=1}^{N} \frac{1}{i!} \frac{\partial^i y(0,q)}{\partial q^i}\Big|_{q=0} q^i, \qquad y_1(0,q) = \lim_{N\to\infty} \sum_{i=0}^{N} \frac{1}{i!} \frac{\partial^i y_1(0,q)}{\partial q^i}\Big|_{q=0} q^i$$

as long as the implicit functions $y(0, q)$ and $y_1(0, q)$ have convergent Taylor series at $q = 0$ with a sufficiently large radius of convergence. In this context, the implicit functions of interest are $y(\bar{z}, 1)$ and $y_1(\bar{z}, 1)$, so a necessary (but not sufficient) condition for nesting is that the radius of convergence of the Taylor series is larger than unity.

Proposition 2. *If* $y(\bar{z}, 1) \neq y(0, 1)$ *or* $y_1(\bar{z}, 1) \neq y_1(0, 1)$*, then risky steady state perturbations are not nested in deterministic steady state perturbations of arbitrary order* N*.*

Proposition 2 provides sufficient conditions for the absence of nesting. Relevant examples where these conditions are satisfied include the small open economy described in section 3 as well as the production economies of section 6. Moreover, even when nesting is possible, the speed of convergence as the order of approximation N increases can be impractically large. For example, in the model of Campbell and Cochrane (1999), figure 1 shows how third-order perturbations fall short of providing a sufficiently accurate approximation of the risky steady state perturbation.

5. Approximate equilibrium risk pricing

This section further inspects the general approximate solution formula (5) applied specifically to the pricing of assets.[14] After having solved for the equilibrium allocation and prices, we can

[13] The online appendix discusses this further by comparing analytically the generalized affine approximation and a third-order perturbation around the deterministic steady state.

[14] This includes welfare costs of economic fluctuations, as they are hold-to-maturity risk premia at the margin (Alvarez and Jermann, 2004; Lopez, 2014).

price assets in zero net supply by relying on the no-arbitrage relation, $0 = \ln E_t^{\mathbb{P}} e^{m_{t+1}+r^j_{t+1}}$, where m is the stochastic discount factor and r^j the return paid off by the jth claim; the expectations operator makes explicit its reference to the physical probability measure ($\mathbb{P}$). While these pricing implications can be included in system (1), we illustrate them separately without loss of generality to make clear how affine approximations help inspecting equilibrium risk prices.

The approximate equilibrium allocation using the method of section 2 includes a state vector $\hat{z}_t \doteq z_t - \widetilde{z} \in \mathbb{R}^{n_z}$ and a log cashflow process $d_t = a_t + S'_d y_t \in \mathbb{R}$, with a_t a trend component with dynamics $a_{t+1} = \mu_d + a_t + C_a \hat{z}_t + D_a(\hat{z}_t)\varepsilon_{t+1}$ and $S_d \in \mathbb{R}^{n_y}$ a selection matrix from the vector of jump variables $y_t \in \mathbb{R}^{n_y}$, whose approximate joint distribution under the physical probability is:

$$\begin{bmatrix} \hat{z}_{t+1} \\ \Delta d_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \mu_d \end{bmatrix} + \begin{bmatrix} A \\ C \end{bmatrix} \hat{z}_t + \begin{bmatrix} B(\hat{z}_t) \\ D(\hat{z}_t) \end{bmatrix} \varepsilon_{t+1}, \qquad \kappa[\alpha(z_t); z_t] \doteq \ln E_t^{\mathbb{P}}[e^{\alpha(z_t)'\varepsilon_{t+1}}],\ \alpha : \mathbb{R}^{n_z} \to \mathbb{R}^{n_\varepsilon} \tag{14}$$

with matrices $A \doteq \widetilde{g}_1 \widetilde{\Psi} + \widetilde{g}_2$, $C \doteq C_a - S'_d \widetilde{\Psi}(I_{n_z} - A)$, $B(\hat{z}_t) \doteq \widetilde{\sigma}_z(z_t)$ and $D(\hat{z}_t) \doteq D_a(\hat{z}_t) + S'_d \widetilde{\Psi} \widetilde{\sigma}_z(z_t)$.

The equilibrium log risk-free rate is described by the Euler equation, $0 = \ln E_t^{\mathbb{P}} e^{m_{t+1}+r_t}$, where the one-period log stochastic discount factor m_t is a linear transformation of the vector of jump, state and trend variables such that $(E_{t+1} - E_t)m_{t+1} = -\gamma(\hat{z}_t)'\varepsilon_{t+1}$, with the price of risk $\gamma(\hat{z}_t) \in \mathbb{R}^{n_\varepsilon}$. It follows that the stochastic discount factor has form:

$$m_{t+1} = -r(\hat{z}_t) - \kappa[-\gamma(\hat{z}_t)'; z_t] - \gamma(\hat{z}_t)'\varepsilon_{t+1} \tag{15}$$

5.1. Asset pricing formulas

Under structure (14) and (15), we are able to characterize in proposition 3 the dependence of generalized equilibrium term-structure components on the states. By proposition 1, proposition 3 also describes equilibrium cashflow strip yields linearized around the risky steady state.

Proposition 3. *Under assumptions* (14) *and* (15)*, we obtain the following linear approximations of yields and risk premia:*

(a) *The nth cashflow strip yield, $y^{(n)}_{d,t} \doteq -\frac{1}{n}\ln(P^{(n)}_{d,t}/D_t)$, with $P^{(n)}_{d,t} = E_t^{\mathbb{P}}(M_{t,t+n}D_{t+n})$ the no-arbitrage price of the nth strip of cashflow process d, has the approximate affine form*

$$y^{(n)}_{d,t} = -\frac{1}{n}A^{(n)} - \frac{1}{n}B^{(n)}_z \hat{z}_t$$

with term structure coefficients determined by the matrix difference equations

$$\begin{aligned} A^{(n)} &= A^{(n-1)} + \mu_d - r(0) - \kappa[-\gamma(0)'; \widetilde{z}] + \kappa[-\gamma(0)' + V_{n-1}(0)'; \widetilde{z}] \\ B^{(n)}_z &= B^{(n-1)}_z A + C - r_1(0) - \kappa_2[-\gamma(0)'; \widetilde{z}] + \kappa_2[-\gamma(0)' + V_{n-1}(0)'; \widetilde{z}] \\ &\quad + \kappa_1[-\gamma(0)'; \widetilde{z}]\gamma_1(0) + \kappa_1[-\gamma(0)' + V_{n-1}(0)'; \widetilde{z}][-\gamma_1(0) + V_{1,n-1}(0)] \end{aligned} \tag{16}$$

with boundary condition $[A^{(0)}; B^{(0)}_z] = 0$, where $V_n(\hat{z}_t)' \doteq D(\hat{z}_t) + B^{(n)}_z B(\hat{z}_t)$ controls the loading of the unexpected component of the nth holding period log return on the shock.

(b) *The holding-period risk premium* $r^{e,(n)}_{d,t+1} \doteq p^{(1)}_{0,t} + p^{(n-1)}_{d,t+1} - p^{(n)}_{d,t}$ *commanded by the n-period ahead cashflow strip is:*

$$\ln E^{\mathbb{P}}_t R^{e,(n)}_{d,t+1} = \kappa[-\gamma(\hat{z}_t)'; z_t] + \kappa[V_{n-1}(\hat{z}_t)'; z_t] - \kappa[-\gamma(\hat{z}_t)' + V_{n-1}(\hat{z}_t)'; z_t]$$

which coincides with negative coentropy $-C_t(M_{t+1}, P^{(n-1)}_{d,t+1})$ *under stochastic discount factor* (15) *and the approximate cashflow strip price.*[15] *This result implies:*

$$r^{e,(n)}_{d,t+1} = \ln E^{\mathbb{P}}_t R^{e,(n)}_{d,t+1} - \kappa[V_{n-1}(\hat{z}_t)'; z_t] + V_{n-1}(\hat{z}_t)'\varepsilon_{t+1}$$

(c) *The per-period hold-to-maturity risk premium commanded by the nth cashflow strip is*

$$\frac{1}{n}\ln E^{\mathbb{P}}_t\left(\frac{P^{(n)}_{0,t} D_{t+n}}{P^{(n)}_{d,t}}\right) = \frac{1}{n}[A^{(n)}_g + A^{(n)}_0 - A^{(n)}_d] + \frac{1}{n}[B^{(n)}_{g,z} + B^{(n)}_{0,n} - B^{(n)}_{d,z}]\hat{z}_t$$

where subscripts 0 *and d index the term structure coefficients associated with real bonds and the relevant cashflow process respectively, and where coefficients* $\{A^{(n)}_g, B^{(n)}_g\}$ *determine the term structure of anticipated cashflow growth* $\frac{1}{n}\ln E^{\mathbb{P}}_t\left(\frac{D_{t+n}}{D_t}\right) = \frac{1}{n}A^{(n)}_g + \frac{1}{n}B^{(n)}_g\hat{z}_t$*:*

$$\begin{aligned} A^{(n)}_g &= \mu_d + A^{(n-1)}_g + \kappa[W_{n-1}(0)'; \overline{z}] \\ B^{(n)}_g &= B^{(n)}_g A + C + \kappa_1[W_{n-1}(0)'; \overline{z}]W_{1,n-1}(0) + \kappa_2[W_{n-1}(0)'; \overline{z}] \end{aligned}$$

with boundary condition $[A^{(0)}_g; B^{(0)}_g] = 0$*, where* $W_n(\hat{z}_t)' \doteq D(\hat{z}_t) + B^{(n)}_g B(\hat{z}_t)$ *controls the loading of the unexpected component of n-period ahead cashflow growth on the shock.*

Appendix C provides a proof of the proposition.

The approximate equilibrium prices of strips characterized by proposition 3 are the basis to price other payoffs. For example, given equilibrium strip prices, the log price-dividend ratio and return on the market portfolio can be constructed as:

$$\ln\left(\frac{P_t}{D_t}\right) = \ln\left(\sum_{n=1}^{\infty} e^{-ny^{(n)}_{d,t}}\right), \qquad E^{\mathbb{P}}_t R^m_{t+1} = \sum_{n=1}^{\infty} \omega_{n,t} E^{\mathbb{P}}_t R^{(n)}_{d,t+1}$$

where $\omega_{n,t} \doteq e^{-ny^{(n)}_{d,t}} / \sum_{n=1}^{\infty} e^{-ny^{(n)}_{d,t}}$ with $\sum_{n=1}^{\infty}\omega_{n,t} = 1$. The approximate distribution of the portfolio can be constructed using simulated moments of strip prices and returns.

More generally, we can characterize the approximate $\mathbb{Q}$-distribution of the state vector and cashflows and how it distorts the $\mathbb{P}$-distribution, as shown in proposition 4.

Proposition 4. *The vector process* $[z; \Delta d]$ *has the approximate ccgf under the physical (*$\mathbb{P}$*) and*

[15] Conditional coentropy of two random variables can be defined as $C_t(e^{x_{t+1}}, e^{y_{t+1}}) \doteq \mathcal{V}_t(e^{x_{t+1}+y_{t+1}}) - \mathcal{V}_t(e^{x_{t+1}}) - \mathcal{V}_t(e^{y_{t+1}})$ (see also Hansen, 2012; Backus et al., 2016).

risk-neutral ($\mathbb{Q}$) probability measures:

$$\begin{aligned}\ln E_t^{\mathbb{P}}[e^{u_z'\hat{z}_{t+1}+u_d\Delta d_{t+1}}] &= u'\mu + \kappa[u'\Sigma(\hat{z}_t); z_t] + u'\Phi\hat{z}_t \\ \ln E_t^{\mathbb{Q}}[e^{u_z'\hat{z}_{t+1}+u_d\Delta d_{t+1}}] &= \ln E_t^{\mathbb{P}}[e^{u_z'\hat{z}_{t+1}+u_d\Delta d_{t+1}}] + \kappa[-\gamma(\hat{z}_t)' + u'\Sigma(\hat{z}_t); z_t] - \kappa[-\gamma(\hat{z}_t)'; z_t] - \kappa[u'\Sigma(\hat{z}_t); z_t]\end{aligned}$$

for $u = [u_z; u_d] \in \mathbb{R}^{n_z+1}$*, where* $\mu \doteq [0_{n_z}; \mu_d] \in \mathbb{R}^{n_z+1}$*,* $\Phi \doteq [A; C] \in \mathbb{R}^{n_z+1\times n_z}$ *and* $\Sigma(\hat{z}_t) \doteq [B(\hat{z}_t); D(\hat{z}_t)] \in \mathbb{R}^{n_z+1\times n_\varepsilon}$.

Appendix D proves this proposition.

5.2. Asset pricing diagnostics

The literature provides a set of diagnostic tools that can be used as a first round of tests of a model of the stochastic discount factor. An important criterion to evaluate the accuracy of an approximation method is that it correctly captures these implications.

A first diagnostic tool to assess a model of the discount factor is the Hansen and Jagannathan (1991) bound, which shows how no-arbitrage pricing implies that the volatility of the discount factor must dominate empirical measures of the maximal risk-return tradeoff. Our affine approximation provides a simple expression for the bound:

$$\left|\frac{E_t^{\mathbb{P}} R_{t+1}^e}{std_t^{\mathbb{P}}(R_{t+1}^e)}\right| \leq \frac{std_t^{\mathbb{P}}(M_{t+1})}{E_t^{\mathbb{P}} M_{t+1}} = \frac{\sqrt{E_t^{\mathbb{P}} e^{2m_{t+1}} - (E_t^{\mathbb{P}} e^{m_{t+1}})^2}}{E_t^{\mathbb{P}} M_{t+1}} \approx \sqrt{e^{\kappa[-2\gamma(\hat{z}_t);z_t]-2\kappa[-\gamma(\hat{z}_t);z_t]} - 1}$$

for all available excess returns. Similar simple expressions for Backus et al. (2014) entropy-based bounds are straightforward to derive.

Our affine approximation also provides tractable expressions for the decompositions of Alvarez and Jermann (2005) and Hansen and Scheinkman (2009) whose properties constitute a diagnostic tool for a model of the discount factor. They show how, under appropriate regularity conditions, the stochastic discount factor can be decomposed as $M_{t+1} = M_{t+1}^P M_{t+1}^T$, where a transient component M_{t+1}^T controls the pricing of long-duration bonds and a martingale component M_{t+1}^P with $E_t^{\mathbb{P}} M_{t+1}^P = 1$ controls the maximum risk premium in the complete-market economy. In absence of a transient component the properties of the martingale component imply a flat term structure of real interest rates; in absence of a martingale component the long-run real bond premium is the highest premium available. Moreover, two main properties of the decomposition that rest on the no-arbitrage pricing formula and Jensen's inequality are (i) the relationship between the transient component of the discount factor and the holding-period return on a infinite-maturity zero-coupon bond,

$$m_{t+1}^T = -\lim_{n\to\infty} r_{0,t+1}^{(n)}$$

and (ii) the property of the entropy ratio,

$$\frac{\mathcal{V}_t^{\mathbb{P}}(M_{t+1}^P)}{\mathcal{V}_t^{\mathbb{P}}(M_{t+1})} = 1 - \frac{E_t^{\mathbb{P}} r_{0,t+1}^{e,(\infty)}}{\mathcal{V}_t^{\mathbb{P}}(M_{t+1})} \geq 1 - \frac{E_t^{\mathbb{P}} r_{0,t+1}^{e,(\infty)}}{\max E_t^{\mathbb{P}} r_{t+1}^e}$$

where the maximum is taken over all available excess returns. These diagnostic properties make clear how an approximation method that correctly captures the term structures of equities and bonds, and especially their long-run properties, is a method that correctly captures this decomposition of pricing kernels and, more specifically, investors' marginal utility. Our emphasis on evaluating the quality of the approximation via the term structures of claims to different cashflow processes in section 6 rests on this observation.

Proposition 5 constructs the approximate decomposition, which Hansen and Scheinkman (2009) show it can be understood using the solution $[\delta; f]$ to the eigenfunction problem:

$$E_t^{\mathbb{P}}[M_{t+1} f(\hat{z}_{t+1})] = \delta f(\hat{z}_t)$$

for some function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and scalar $\delta \in \mathbb{R}$, with transient and martingale components constructed as $M_{t+1}^T = \delta f(\hat{z}_t)/f(\hat{z}_{t+1})$ and $M_{t+1}^P = M_{t+1} f(\hat{z}_{t+1})/\delta f(\hat{z}_t)$.

Proposition 5. *In the context of the general model of section 2, the approximate generalized affine solution of the eigenfunction problem is $\delta \in \mathbb{R}$ and $f(\hat{z}_t) = e^{u_z' \hat{z}_t}$ with:*

$$\begin{aligned} 0 &= \ln(\delta) - r(0) + \kappa[-\gamma(0)' + V(0)'; \widetilde{z}] - \kappa[-\gamma(0)'; \widetilde{z}] \\ u_z' &= u_z' A \hat{z}_t - r_1(0)\hat{z}_t - \kappa_2[-\gamma(0); \widetilde{z}] + \kappa_2[-\gamma(0) + V(0); \widetilde{z}] \\ &\quad + \kappa_1[\gamma(0)'; \widetilde{z}]\gamma_1(0) + \kappa_1[-\gamma(0)' + V(0)'; \widetilde{z}][-\gamma(0)' + V(0)'] \end{aligned}$$

where $V(\hat{z}_t)' \doteq u_z' B(\hat{z}_t)$.

Appendix E provides a proof of the proposition. Appendix F shows the equivalence of Hansen-Scheinkman and Alvarez-Jermann decompositions in this context of affine pricing.

5.3. Inspecting the mechanism

Affine approximations are particularly useful to provide an intuitive understanding of the macroeconomic forces that drive the prices of financial claims and the risk premia they command. Using equation $r_{t+1}^{e,(n)} = E_t^{\mathbb{P}} r_{t+1}^{e,(n)} + V_{n-1}(\hat{z}_t)'\varepsilon_{t+1}$ from proposition 3, we can attribute the coentropy of the quantity of risk in the nth strip return with a shock along direction $\alpha(\hat{z}_t)$ to two components,

$$-C_t\left(\exp[\alpha(\hat{z}_t)'\varepsilon_{t+1}], \exp\left[\Big(\underbrace{D(\hat{z}_t)}_{\substack{\text{short-run} \\ \text{cashflow risk}}} + \underbrace{B_z^{(n-1)} B(\hat{z}_t)}_{\substack{\text{long-run cashflow} \\ \text{and discount-rate risk}}} \Big)\varepsilon_{t+1}\right]\right) \qquad (17)$$

which are the basis to understand the shape of the term structure of holding-period risk premia when $\alpha(\hat{z}_t) = -\gamma(\hat{z}_t)$. The first element on the right-hand side of the equation controls the cashflow effect due to contemporaneous shocks to dividends. The second element captures income and substitution effects of past shocks as well as the role of each state variable in shaping the term structure.

Another way the generalized affine approximation facilitates an intuitive understanding of the asset-pricing implications of the macro-finance model is by providing simple expressions for the dynamic value decomposition proposed by Borovicka and Hansen (2014) as measures to quantify the exposures of cashflows to shocks over alternative horizons and the corresponding compensations commanded by investors. In particular, for any marginal increase in one-step ahead uncertainty along dimension $\alpha(\hat{z}_t)$ we can define cashflow and discount-rate elasticities as:

$$\begin{aligned}
\varepsilon_{g,t}^{(n)} &\doteq \frac{d}{dr}\ln E_t^{\mathbb{P}}\left[\frac{D_{t+n}}{D_t}e^{r\alpha(\hat{z}_t)'\varepsilon_{t+1}-\kappa[r\alpha(\hat{z}_t)';z_t]}\right]_{r=0} = \kappa_1[D(\hat{z}_t)+B_g^{(n-1)}B(\hat{z}_t);z_t]\alpha(\hat{z}_t) \\
\varepsilon_{p,t}^{(n)} &\doteq \frac{d}{dr}\ln E_t^{\mathbb{P}}\left[\frac{D_{t+n}}{D_t}e^{r\alpha(\hat{z}_t)'\varepsilon_{t+1}-\kappa[r\alpha(\hat{z}_t)';z_t]}\right]_{r=0} - \frac{d}{dr}\ln E_t^{\mathbb{P}}\left[M_{t,t+n}\frac{D_{t+n}}{D_t}e^{r\alpha(\hat{z}_t)'\varepsilon_{t+1}-\kappa[r\alpha(\hat{z}_t)';z_t]}\right]_{r=0} \\
&= \kappa_1[D(\hat{z}_t)+B_g^{(n-1)}B(\hat{z}_t);z_t]\alpha(\hat{z}_t) - \kappa_1[-\gamma(\hat{z}_t)+D(\hat{z}_t)+B_z^{(n-1)}B(\hat{z}_t);z_t]\alpha(\hat{z}_t)
\end{aligned} \tag{18}$$

Appendix F derives expression (18). These elasticities capture the impact of current shocks on future cashflows ($\varepsilon_{g,t}^{(n)}$) and on future expected returns ($\varepsilon_{p,t}^{(n)}$), while the impact on valuations can be recovered as the value elasticity $\varepsilon_{g,t}^{(n)} - \varepsilon_{p,t}^{(n)}$.

6. Applications and accuracy of the approximation

This section illustrates the performance of our generalized affine approximation by applying it to a few challenging models. Models with Campbell-Cochrane habits are particularly suited to test our approximation as they display strong heteroskedasticity; the state of the economy is driven by consumption news, which are endogenous objects outside an endowment economy. Models with time-varying disaster risk and recursive utility similarly produce variation in risk premia, while non-Gaussianities make loglinear-lognormal methods inapplicable.

To clarify the importance of our extension of extant affine approximations we compare four solution methods. First, projected solutions reflect accurate numerical procedures that we take as the exact solution.[16] Second, we report our generalized affine approximation. Third, we report a linear perturbation around the risky steady state, which coincides with our affine approximation except for its treatment of innovations of the state vector. Finally, we report the loglinear-lognormal approximation in Malkhozov (2014), when applicable.

We evaluate the quality of our approximation by comparing the term structures of zero-coupon equities and, in production economies, multiperiod Euler equation errors; this exercise allows for decomposing the quality of the approximation at different time horizons and for claims that are the basis for pricing other more complex assets. We define errors in the n-period Euler equation from a solution for consumption $c^{(0)}(z_t)$ as:

$$EEE^{(n)}(z_t) \doteq \log_{10}\left|1 - e^{c^{(n)}(z_t)-c^{(0)}(z_t)}\right|$$

where $c_t^{(n)}(z_t)$ solves equation $0 = \ln E_t e^{m_{t+1}[c_{t+1}^{(n-1)}(z_t),c_t^{(n)}(z_t)]+r_t}$, for points z_t that cover a high-probability region of the state space, and a stochastic discount factor m_{t+1} that is a function of consumption. A n-period Euler equation error of $-\varepsilon$ implies that the consumer is making a one dollar mistake

[16] Global solutions are projected onto the subspace spanned by a polynomial basis collocated on a grid of Chebyshev points. When the state space is multidimensional we rely on collocation on (adaptive, anisotropic) Smolyak sparse grids (Judd et al., 2014).

in how much she decides to save over a n-period horizon for every 10^{ε} dollars spent. Since errors accumulate as the horizon increase, multiperiod Euler equation errors provide an indication of how good the approximation is for long-term valuations.

6.1. Application 1: Campbell and Cochrane (1999)

Figure 6 complements figure 1 by comparing the global solution with our proposed generalized affine solution and the alternative affine approximations. The figure reports the term structure of equilibrium risk premia and realized return volatilities of zero-coupon equities and bonds. While figure 1 described the approximation of the map of the state space into equilibrium prices, figure 6 is based on simulations and hence it reflects both the approximation of the map *and* the approximation of the dynamics of state variables.

In this context, since the unique state is exogenous in this endowment economy, standard and generalized affine approximations coincide (see footnote 10). Moreover, affine approximations are preferable to a linear perturbation around the risky steady state because they do not induce spurious dynamics in the state vector via the approximation of function $\Lambda(\hat{s}_t)$; an approximation that simulates using linearized conditional volatilities overstates the asymptotic risk premium, thereby distorting the representation of investors' marginal utility of wealth.

Relative to the projected solution, the fit of the generalized affine approximation manages to capture the level, amplitude and shape of the term structures.

6.2. Application 2: Jermann (1998) with nonlinear habits

A representative consumer with Campbell-Cochrane habits in consumption lives in a production economy and chooses output $Y_t = A_t^{1-\alpha} K_t^{\alpha}$ and the trajectory of capital, whose accumulation is subject to adjustment costs:

$$K_{t+1} = \left[1 - \delta + \Phi\left(\frac{I_t}{K_t}\right)\right] K_t = e^{\mu} K_t + \frac{\bar{\imath}}{1 - \frac{1}{\xi}} \left[\left(\frac{I_t}{\bar{\imath} K_t}\right)^{1-\frac{1}{\xi}} - 1\right] K_t$$

where $\bar{\imath} \doteq \frac{\delta}{1+1/\xi}$ is the deterministic steady-state investment-capital ratio. Output is devoted to consumption or to investment, $Y_t = C_t + I_t$. Technology and habits are driven by:

$$\begin{aligned} a_{t+1} &= \mu + a_t + \sigma \varepsilon_{t+1} \\ s_{t+1} &= (1 - \phi)s + \phi s_t + \Lambda(\hat{s}_t)(c_{t+1} - E_t c_{t+1}) \end{aligned}$$

where $\varepsilon_t \sim Niid(0, 1)$.

Joint optimality of consumption, investment and capital accumulation imply conditions:

$$\left(\frac{I_t}{\bar{\imath} K_t}\right)^{\frac{1}{\xi}} K_{t+1} = E_t \beta \left(\frac{C_{t+1} S_{t+1}}{C_t S_t}\right)^{-\gamma} W_{t+1} \qquad W_t \doteq D_t + \left(\frac{I_t}{\bar{\imath} K_t}\right)^{\frac{1}{\xi}} K_{t+1} \tag{19}$$

$$\frac{C_t}{K_t} = \left(\frac{A_t}{K_t}\right)^{1-\alpha} - \frac{I_t}{K_t} \qquad D_t \doteq \alpha C_t - (1 - \alpha) I_t \tag{20}$$

and terminal condition $\lim_{h\to\infty} E_t\beta^h(\frac{C_{t+h}S_{t+h}}{C_tS_t})^{-\gamma}(\frac{I_{t+h}}{\bar{i}K_{t+h}})^{1/\xi} = 0$. When combined with the terminal condition we can rewrite forward-looking difference equation (19) as:

$$\left(\frac{I_t}{\bar{i}K_t}\right)^{\frac{1}{\xi}} K_{t+1} = \sum_{n=1}^{\infty} P_{d,t}^{(n)}, \qquad P_{d,t}^{(n)} = E_t\beta\left(\frac{C_{t+1}S_{t+1}}{C_tS_t}\right)^{-\gamma} P_{d,t+1}^{(n-1)}, \quad P_{d,t}^{(0)} = D_t \tag{21}$$

We rewrite structural equations (19) and (20) in log-form (1):

$$0 = \ln E_t \exp[f(y_t, z_t, y_{t+1}, z_{t+1})]$$

$$f(y_t, z_t, y_{t+1}, z_{t+1}) = \begin{bmatrix} \ln(\beta) - \gamma(\Delta ck_{t+1} + \Delta ka_{t+1} + \mu + \sigma\varepsilon_{t+1}) - \gamma\Delta s_{t+1} - \frac{1}{\xi}[ik_t - \ln(\bar{i})] + w_{t+1} \\ \ln(e^{ck_t} + e^{ik_t}) + (1-\alpha)ka_t \\ \xi w_t - \xi\ln\left[\alpha e^{ck_t} + \left(\alpha - 1 + \frac{1}{1-\frac{1}{\xi}}\right)e^{ik_t} + \left(e^{\mu} - \frac{\bar{i}}{1-\frac{1}{\xi}}\right)\bar{i}^{-\frac{1}{\xi}}e^{\frac{1}{\xi}ik_t}\right] \end{bmatrix}$$

$$z_{t+1} = \begin{bmatrix} ka_t + \ln\left(1 + e^{-\mu}\bar{i}\frac{e^{(1-1/\xi)[ik_t - \ln(\bar{i})]} - 1}{1-1/\xi}\right) \\ \phi\hat{s}_t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \Lambda(\hat{s}_t) \\ 0 \end{bmatrix}(ck_{t+1} - E_t ck_{t+1}) + \begin{bmatrix} -\sigma \\ 0 \\ 1 \end{bmatrix}\varepsilon_{t+1}$$

with the state vector $z_t = [ka_t; \hat{s}_t; \varepsilon_t]$ and the vector of decision variables $y_t = [ck_t; ik_t; w_t]$, where $ka_t \doteq k_t - a_t$, $ck_t \doteq c_t - k_t$ and $ik_t \doteq i_t - k_t$.

6.2.1. Affine approximation

We use the algorithm in section 2 to obtain a linearization of the equilibrium conditions. While the correct approximation would work with summation specification (21), we work with difference specification (19) and update subsequently the solution using equation (21).

Step 1. Write expectational equations in terms of a certainty-equivalent and an entropy terms:

$$0 = \xi\ln(\beta e^{-\gamma\mu}) - \gamma\xi E_t\Delta(ck_{t+1} - ka_{t+1}) + \gamma\xi(1-\phi)\hat{s}_t - ik_t + \ln(\bar{i}) + \xi E_t w_{t+1} + \xi\mathcal{V}_t\left(e^{-\gamma[1+\Lambda(\hat{s}_t)]ck_{t+1} + w_{t+1}}\right)$$

Step 2. Conjecture an affine solution in the state vector and use it to characterize the entropy term: $ck_t = \widetilde{ck} + \widetilde{\psi}_{ck}\hat{ka}_t + \widetilde{\psi}_{cs}\hat{s}_t$, $ik_t = \widetilde{ik} + \widetilde{\psi}_{ik}\hat{ka}_t + \widetilde{\psi}_{is}\hat{s}_t$, $w_t = \widetilde{w} + \widetilde{\psi}_{wk}\hat{ka}_t + \widetilde{\psi}_{ws}\hat{s}_t$, hence $c_{t+1} - E_t c_{t+1} = \frac{-\widetilde{\psi}_{ck}}{1-\widetilde{\psi}_{cs}\Lambda(\hat{s}_t)}\sigma\varepsilon_{t+1}$, $w_{t+1} - E_t w_{t+1} = -\left(\widetilde{\psi}_{wk} + \widetilde{\psi}_{ws}\frac{\widetilde{\psi}_{ck}\Lambda(\hat{s}_t)}{1-\widetilde{\psi}_{cs}\Lambda(\hat{s}_t)}\right)\sigma\varepsilon_{t+1}$, and

$$\mathcal{V}_t\left(e^{-\gamma[1+\Lambda(\hat{s}_t)]ck_{t+1} + w_{t+1}}\right) = \left(\frac{-\gamma\widetilde{\psi}_{ck}[1+\Lambda(\hat{s}_t)]}{1-\widetilde{\psi}_{cs}\Lambda(\hat{s}_t)} + \widetilde{\psi}_{wk} + \widetilde{\psi}_{ws}\frac{\widetilde{\psi}_{ck}\Lambda(\hat{s}_t)}{1-\widetilde{\psi}_{cs}\Lambda(\hat{s}_t)}\right)^2\frac{\sigma^2}{2}$$

Step 3. Linearize and match coefficients or, equivalently, solve matrix equation (5).

6.2.2. *Updated affine approximation*

We can update the solution for the investment-capital ratio via summation specification (21), where dividends are defined as the marginal product of capital net of new investment, with:

$$d_t = k_t + \widetilde{dk} + \widetilde{\psi}_{dk}\hat{ka}_t + \widetilde{\psi}_{ds}\hat{s}_t, \qquad \widetilde{dk} = \ln\left(\alpha e^{\widetilde{ck}} - (1-\alpha)\bar{i}\right)$$

$$\widetilde{\psi}_{dz} = \frac{\alpha e^{\widetilde{ck}}}{\alpha e^{\widetilde{ck}} - (1-\alpha)\bar{i}}\widetilde{\psi}_{cz} - \frac{(1-\alpha)\bar{i}}{\alpha e^{\widetilde{ck}} - (1-\alpha)\bar{i}}\widetilde{\psi}_{iz}, \quad z = k, s$$

Given the affine approximation for ck_t and ik_t we can approximate equilibrium yields as:

$$\begin{aligned}
pd_t^{(n)} &= A^{(n)} + B_k^{(n)}\hat{ka}_t + B_s^{(n)}\hat{s}_t \\
&= \ln(\widetilde{\beta}) + \left(B_s^{(n-1)}\phi + (1-\phi)[\gamma(1+\psi_{ck}) - \psi_{ds}]\right)\hat{s}_t + [1 + \psi_{dk} - \gamma(1+\psi_{ck})][(\psi_{kk}-1)\hat{\hat{k}}_t + \psi_{ks}\hat{s}_t] \\
&\quad + B_k^{(n-1)}(\widetilde{\psi}_{kk}\hat{ka}_t + \widetilde{\psi}_{ks}\hat{s}_t) + \frac{1}{2}V_0^{(n-1)} + \frac{1}{2}V_1^{(n-1)}\hat{s}_t \\
V_0^{(n-1)} &\doteq \left(\frac{(\widetilde{\psi}_{ds} + B_s^{(n-1)} - \gamma)\Lambda(0) - \gamma}{1 - \widetilde{\psi}_{cs}\Lambda(0)}\widetilde{\psi}_{ck} + (\widetilde{\psi}_{dk} + B_k^{(n-1)})\right)^2 \sigma^2 \\
V_1^{(n-1)} &\doteq \frac{\partial}{\partial\hat{s}_t}\left(\frac{(\widetilde{\psi}_{ds} + B_s^{(n-1)} - \gamma)\Lambda(0) - \gamma}{1 - \widetilde{\psi}_{cs}\Lambda(0)}\widetilde{\psi}_{ck} + (\widetilde{\psi}_{dk} + B_k^{(n-1)})\right)^2 \sigma^2
\end{aligned}$$

and identify the affine approximation by matching coefficients.

Therefore, the updated approximate solution for the investment-capital ratio is:

$$\begin{aligned}
ik_t^* &= \ln(\bar{i}) + \xi\left[d_t - k_t - \Delta k_{t+1} + \ln\left(\sum_{n=1}^{\infty} e^{A^{(n)} + B_k^{(n)}\hat{ka}_t + B_s^{(n)}\hat{s}_t}\right)\right] \\
&= \ln(\bar{i}) + \xi\left[\widetilde{dk} + \widetilde{\psi}_{dk}\hat{ka}_t + \widetilde{\psi}_{ds}\hat{s}_t - \ln\left(e^{\mu} + \bar{i}\frac{e^{\left(1-\frac{1}{\xi}\right)(\widetilde{\psi}_{ik}\hat{ka}_t + \widetilde{\psi}_{is}\hat{s}_t)} - 1}{1 - \frac{1}{\xi}}\right) + \ln\left(\sum_{n=1}^{\infty} e^{A^{(n)} + B_k^{(n)}\hat{ka}_t + B_s^{(n)}\hat{s}_t}\right)\right]
\end{aligned}$$

6.2.3. *Quality of the approximation*

We calibrate the model using the values listed in table 2. Habit-related parameters are the same as in Campbell and Cochrane (1999) but at quarterly frequency; technology implies the same average consumption growth and volatility of consumption innovations in Campbell and Cochrane (1999). The additional parameters are set as in Jermann (1998) and imply a steady-state investment-capital ratio of 10% on a yearly basis and a detrended investment process that is twice as volatile as detrended consumption.

Figures 2 compares the global solution to our affine approximation. Similarly to figure 1, the non-updated affine approximation makes a reasonably good job at solving the model. However, we can increase substantially the quality of the approximation by updating the solution; the updated affine approximation is nearly equivalent to the global solution.

Figure 7 plots term structure implications; in this context, all risk-adjusted linearizations perform similarly, as people absorb shocks through investment to stabilize consumption and hence display

Parameter	value
Capital share, α	0.36
Investment-capital ratio, $\bar{\imath} = \frac{\delta}{1+1/\xi}$	0.025
Capital adjustment cost curvature, $\frac{1}{\xi}$	0.23
Discount factor, β	0.9746
Utility curvature, γ	2
Habit persistence, ϕ	$0.87^{1/4}$
Mean TFP growth (in %), μ	1.89/4
Standard deviation of TFP innovations (in %), σ	1.17
implied standard deviation of consumption innovations (in %)	$1.50/\sqrt{4}$

Table 2: Deep parameters and their calibration (quarterly frequency) in the RBC model with Campbell-Cochrane habits.

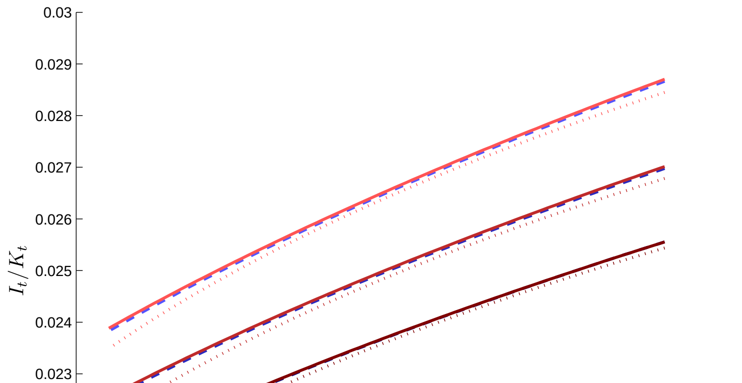

Figure 2: Comparison of affine and projected solutions to the maps of states into the investment-capital in the model with capital accumulation and Campbell-Cochrane habits. Projections (dashed blue lines), updated affine approximation (solid red lines) and non-updated affine approximation (dotted red lines). The projected solution uses Smolyak collocation of Chebyshev polynomials of up to degree 8 and 5-point Gauss-Hermite quadrature. The state space consists of surplus consumption (S) and detrended capital (K/A). The grid for S covers the entire simulated distribution of surplus consumption; the three values of K/A represent minimum, maximum and mean values of detrended capital.

a substantially lower risk aversion than in the endowment economy of Campbell and Cochrane (1999).

Figure 3 shows multiperiod Euler equation errors. The accuracy of our global solution in terms of conventional 1-step ahead Euler equation errors is consistently lower than −2, and remains with maximums of around −2 over arbitrarily long horizons. These values are considerably lower than under the global solution but remain relatively small; values of around −3 are typically retained as acceptable in the extant literature.

6.2.4. Application 3: Lopez et al. (2015)

A representative consumer with external habits in market and home consumption:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{(C_t - X_t^c)^{1-\gamma} - 1}{1-\gamma} + \chi \left(\frac{S}{Z}\right)^{-\gamma} \frac{(H_t - X_t^h)^{1-\gamma} - 1}{1-\gamma} \right]$$

lives in a production economy with monopolistically competitive firms subject to Calvo price rigidities. Home consumption is produced with technology $H_t = A_t(1 - N_t)$, where $N_t \in [0, 1]$ denotes hours worked. $S \doteq 1 - X^c/C$ and $Z \doteq 1 - X^h/H$ define surplus market and home consumption levels. There is no capital accumulation and a continuum of firms $i \in (0, 1)$ produce according to technology $Y_{i,t} = A_t N_{i,t}^{1-\alpha}$, sell goods at price $P_{i,t}$ and face the demand curve $Y_{i,t} = (P_{i,t}/P_t)^{-\varepsilon} Y_t$. Output aggregates as $Y_t = A_t N_t^{1-\alpha} \nu_t^{-(1-\alpha)}$, where $\nu_t \doteq \int (P_{i,t}/P_t)^{-\varepsilon/(1-\alpha)} di$ has dynamics:

$$\nu_t = \eta e^{\frac{\varepsilon}{1-\alpha}\pi_t} \nu_{t-1} + (1-\eta) \left(\frac{1 - \eta e^{(\varepsilon-1)\pi_t}}{1-\eta} \right)^{\frac{\varepsilon}{(1-\alpha)(\varepsilon-1)}}$$

Output is devoted entirely to consumption $C_t = Y_t$. Monetary policy follows a Taylor rule for the interest rate $i_t = i^* + \phi_\pi(\pi_t - \pi^*)$; fiscal policy provides an employment subsidy τ^f levied in lump-sum fashion on firms to align risky and deterministic steady-state values of employment.

The necessary conditions for a competitive equilibrium include the Euler equation:

$$0 = \ln E_t e^{\ln(\beta) - \gamma \Delta c_{t+1} - \gamma \Delta s_{t+1} - \pi_{t+1} + i_t}$$

optimal labor supply and demand curves:

$$\begin{aligned} w_t - p_t &= \ln(\chi) + \gamma c_t - \gamma h_t + a_t - \gamma \xi_2 \hat{s}_t \\ w_t - p_t &= mc_t - \ln(1 - \tau^f) + \ln\left(\frac{\partial Y_t}{\partial N_t}\right) \end{aligned}$$

where $w_t - p_t$ denotes the real wage rate and mc_t firms' marginal costs; and optimal price setting:

$$\begin{aligned} e^{\ell_t} &= E_t e^{\ln(\beta\eta) + \frac{\varepsilon}{1-\alpha}\pi_{t+1} + (1-\gamma)\Delta a_{t+1} + \ell_{t+1}} + e^{(1-\gamma)\widetilde{c}_t - \gamma \hat{s}_t + mc_t + \ln[\varepsilon/(\varepsilon-1)] - \ln(\nu_t)} \\ e^{\ell_t} \left(\frac{1 - \eta e^{(\varepsilon-1)\pi_t}}{1-\eta} \right)^{\frac{1-\alpha+\alpha\varepsilon}{(1-\alpha)(\varepsilon-1)}} &= E_t e^{\ln(\beta\eta) + (\varepsilon-1)\pi_{t+1} + (1-\gamma)\Delta a_{t+1} + \ell_{t+1}} \left(\frac{1 - \eta e^{(\varepsilon-1)\pi_{t+1}}}{1-\eta} \right)^{\frac{1-\alpha+\alpha\varepsilon}{(1-\alpha)(\varepsilon-1)}} + e^{(1-\gamma)\widetilde{c}_t - \gamma \hat{s}_t} \end{aligned}$$

where $\widetilde{c}_t \doteq c_t - a_t$ is an output gap measure, π_t is the inflation rate and ℓ_t is an auxiliary variable. Finally, surplus consumption and technology are driven by:

$$\begin{aligned}
\hat{s}_{t+1} &= \rho_s \hat{s}_t + \Lambda(\hat{s}_t)(c_{t+1} - E_t c_{t+1}) \\
a_{t+1} &= \mu + a_t + u_t + \sigma\varepsilon_{t+1}, \quad u_{t+1} = \rho_u u_t - \phi\sigma\varepsilon_{t+1}
\end{aligned}$$

with $\varepsilon_t \sim Niid(0, 1)$. We rewrite the optimality conditions in log form (1) as:

$$0 = \ln E_t \exp[f(y_t, z_t, y_{t+1}, z_{t+1})]$$

$$f(y_t, z_t, y_{t+1}, z_{t+1}) = \begin{bmatrix}
\ln(\beta) - \gamma\mu + i^* - \widetilde{\pi} + \phi_\pi \hat{\pi}_t - \gamma(\Delta\hat{\widetilde{c}}_{t+1} + u_t + \sigma\varepsilon_{t+1}) - \gamma\Delta\hat{s}_{t+1} - \hat{\pi}_{t+1} \\
\ln(\beta e^{(1-\gamma)\mu}\eta\widetilde{\Pi}^{\frac{\varepsilon}{1-\alpha}}) + \frac{\varepsilon}{1-\alpha}\hat{\pi}_{t+1} + (1-\gamma)(u_t + \sigma\varepsilon_{t+1}) + \Delta\hat{\ell}_{t+1} - w_{1,t} \\
\ln(\beta e^{(1-\gamma)\mu}\eta\widetilde{\Pi}^{\varepsilon-1}) + (\varepsilon - 1)\hat{\pi}_{t+1} + (1-\gamma)(u_t + \sigma\varepsilon_{t+1}) + \Delta\hat{\ell}_{t+1} + \Delta\hat{w}_{3,t+1} - w_{2,t} \\
w_{1,t} - \ln\left(1 - \frac{\chi\varepsilon(1-\tau)}{(1-\alpha)(\varepsilon-1)} \frac{\widetilde{N}\widetilde{\nu}^{-1}}{e^{\widetilde{\ell}}} e^{\frac{1}{1-\alpha}\hat{\widetilde{c}}_t - \gamma(1+\xi_2)\hat{s}_t - \hat{\ell}_t} \left[1 - \widetilde{N}\hat{\nu}_t e^{\frac{1}{1-\alpha}\hat{\widetilde{c}}_t}\right]^{-\gamma}\right) \\
w_{2,t} - \ln\left(1 - \frac{(\widetilde{N}\widetilde{\nu}^{-1})^{(1-\gamma)(1-\alpha)}}{e^{\widetilde{\ell}+\widetilde{w}_3}} e^{(1-\gamma)\hat{\widetilde{c}}_t - \gamma\hat{s}_t - \hat{\ell}_t - \hat{w}_{3,t}}\right) \\
w_{3,t} - \frac{1-\alpha+\alpha\varepsilon}{(1-\alpha)(\varepsilon-1)} \ln\left(\frac{1-\eta\widetilde{\Pi}^{\varepsilon-1}e^{(\varepsilon-1)\hat{\pi}_t}}{1-\eta}\right) \\
\frac{\varepsilon}{1-\alpha}\pi_{t+1} - x_t^\pi \\
\frac{\varepsilon}{(1-\alpha)(\varepsilon-1)} \ln\left(\frac{1-\eta\widetilde{\Pi}^{\varepsilon-1}e^{(\varepsilon-1)\hat{\pi}_{t+1}}}{1-\eta}\right) - x_t^\nu \\
e^{\frac{\varepsilon}{1-\alpha}\pi_t} - w_t^\pi \\
e^{\frac{\varepsilon}{(1-\alpha)(\varepsilon-1)} \ln\left(\frac{1-\eta\widetilde{\Pi}^{\varepsilon-1}e^{(\varepsilon-1)\hat{\pi}_t}}{1-\eta}\right)} - w_t^\nu
\end{bmatrix}$$

for the vector of decision variables is $y_t = [\hat{\widetilde{c}}_t; \hat{\pi}_t; \hat{\ell}_t; w_{1,t}; w_{2,t}; w_{3,t}; x_t^\pi; x_t^\nu; w_t^\pi; w_t^\nu]$ and with state variables $z_t = [u_t; \hat{s}_t; \nu_t]$:

$$\begin{bmatrix} u_{t+1} \\ \hat{s}_{t+1} \\ \nu_{t+1} \end{bmatrix} = \begin{bmatrix} \rho_u u_t \\ \rho_s \hat{s}_t \\ \eta\widetilde{\Pi}^{\frac{\varepsilon}{1-\alpha}} \nu_t e^{x_t^\pi} + (1-\eta)e^{x_t^\nu} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Lambda(\hat{s}_t) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta\nu_t & 1-\eta \end{bmatrix} (y_{t+1} - E_t y_{t+1}) + \begin{bmatrix} -\phi \\ \Lambda(\hat{s}_t) \\ 0 \end{bmatrix} \sigma\varepsilon_{t+1}$$

Auxiliary variables $[w_{1,t}; w_{2,t}; w_{3,t}; x_t^\pi; x_t^\nu; w_t^\pi; w_t^\nu]$ are necessary to cast the model in form (1).

6.2.5. *Affine approximation*

We use the algorithm in section 2 to obtain a linearization of the equilibrium conditions. Note that the auxiliary variable ℓ_t is described by a forward-looking difference equation that, when combined with the appropriate terminal condition, unwinds into an infinite summation of auxiliary variables. While the correct approximation would work with summation specification, we work with the difference specification and update subsequently the solution.

Step 1. Write expectational equations in terms of a certainty-equivalent and an entropy terms:

$$
\begin{aligned}
0 &= \ln(\beta) - \gamma\mu + i^* - \widetilde{\pi} + \phi_\pi \hat{\pi}_t - \gamma(E_t \Delta\hat{\widetilde{c}}_{t+1} + u_t) + \gamma(1-\rho_s)\hat{s}_t - E_t\hat{\pi}_{t+1} + \mathcal{V}_t\left(e^{-x_t(\widetilde{c}_{t+1}+\sigma\varepsilon_{t+1})-\pi_{t+1}}\right)\\
0 &= \ln(\beta e^{(1-\gamma)\mu}\eta\widetilde{\Pi}^{\frac{\varepsilon}{1-\alpha}}) + \frac{\varepsilon}{1-\alpha}E_t\hat{\pi}_{t+1} + (1-\gamma)u_t + E_t\Delta\hat{\ell}_{t+1} - w_{1,t} + \mathcal{V}_t\left(e^{\frac{\varepsilon}{1-\alpha}\pi_{t+1}+(1-\gamma)\sigma\varepsilon_{t+1}+\ell_{t+1}}\right)\\
0 &= \ln(\beta e^{(1-\gamma)\mu}\eta\widetilde{\Pi}^{\varepsilon-1}) + (\varepsilon-1)E_t\hat{\pi}_{t+1} + (1-\gamma)u_t + E_t\Delta\hat{\ell}_{t+1} + E_t\Delta\hat{w}_{3,t+1} - w_{2,t} + \mathcal{V}_t\left(e^{(\varepsilon-1)\pi_{t+1}+(1-\gamma)\sigma\varepsilon_{t+1}+\ell_{t+1}+w_{3,t+1}}\right)\\
0 &= \frac{\varepsilon}{1-\alpha}E_t\pi_{t+1} + \mathcal{V}_t\left(e^{\frac{\varepsilon}{1-\alpha}\pi_{t+1}}\right) - x_t^\pi\\
0 &= \frac{\varepsilon}{1-\alpha+\alpha\varepsilon}E_t w_{3,t+1} + \mathcal{V}_t\left(e^{\frac{\varepsilon}{1-\alpha+\alpha\varepsilon}w_{3,t+1}}\right) - x_t^\nu
\end{aligned}
$$

Step 2. Conjecture the approximate solution $\widetilde{c}_t = \widetilde{c} + \psi_{cu}u_t + \psi_{cs}\hat{s}_t + \psi_{c\nu}\hat{\nu}_t$, $\pi_t = \widetilde{\pi} + \psi_{\pi u}u_t + \psi_{\pi s}\hat{s}_t + \psi_{\pi\nu}\hat{\nu}_t$ and $\ell_t = \widetilde{\ell} + \psi_{\ell u}u_t + \psi_{\ell s}\hat{s}_t + \psi_{\ell\nu}\hat{\nu}_t$, and characterize the entropy terms. We use:

$$
\begin{bmatrix} c_{t+1} - E_t c_{t+1} \\ \pi_{t+1} - E_t\pi_{t+1} \end{bmatrix} = \begin{bmatrix} \widetilde{\sigma}_c(z_t) \\ \widetilde{\sigma}_\pi(z_t) \end{bmatrix}\varepsilon_{t+1}, \qquad \begin{bmatrix} \widetilde{\sigma}_c(z_t) \\ \widetilde{\sigma}_\pi(z_t) \end{bmatrix} \doteq \begin{bmatrix} 1-\psi_{cs}\Lambda(\hat{s}_t) & -\psi_{c\nu}\Upsilon(\nu_t) \\ -\psi_{\pi s}\Lambda(\hat{s}_t) & 1-\psi_{\pi\nu}\Upsilon(\nu_t) \end{bmatrix}^{-1} \begin{bmatrix} 1-\psi_{cu}\phi \\ -\psi_{\pi u}\phi \end{bmatrix}\sigma
$$

and hence:

$$
\begin{aligned}
\mathcal{V}_t\left(e^{-x_t(\widetilde{c}_{t+1}+\sigma\varepsilon_{t+1})-\pi_{t+1}}\right) &= \frac{1}{2}\left[\frac{\gamma}{S}\sqrt{1-2\hat{s}_t}\widetilde{\sigma}_c(\hat{z}_t) + \widetilde{\sigma}_\pi(\hat{z}_t)\right]^2 && \approx \mathcal{V}_{0,1} + \mathcal{V}_{s,1}\hat{s}_t + \mathcal{V}_{\nu,1}\hat{\nu}_t\\
\mathcal{V}_t\left(e^{\frac{\varepsilon}{1-\alpha}\pi_{t+1}+(1-\gamma)\sigma\varepsilon_{t+1}+\ell_{t+1}}\right) &= \frac{1}{2}\left[\frac{\varepsilon}{1-\alpha}\widetilde{\sigma}_\pi(\hat{z}_t) + (1-\gamma)\sigma + \widetilde{\sigma}_\ell(\hat{z}_t)\right]^2 && \approx \mathcal{V}_{0,2} + \mathcal{V}_{s,2}\hat{s}_t + \mathcal{V}_{\nu,2}\hat{\nu}_t\\
\mathcal{V}_t\left(e^{(\varepsilon-1)\pi_{t+1}+(1-\gamma)\sigma\varepsilon_{t+1}+\ell_{t+1}+w_{3,t+1}}\right) &= \frac{1}{2}\left[\frac{(\varepsilon-1)(1-\alpha)-\varepsilon\eta\widetilde{\Pi}^{\varepsilon-1}}{1-\alpha}\widetilde{\sigma}_\pi(\hat{z}_t) + (1-\gamma)\sigma + \widetilde{\sigma}_\ell(\hat{z}_t)\right]^2 && \approx \mathcal{V}_{0,3} + \mathcal{V}_{s,3}\hat{s}_t + \mathcal{V}_{\nu,3}\hat{\nu}_t\\
\mathcal{V}_t\left(e^{\frac{\varepsilon}{1-\alpha}\pi_{t+1}}\right) &= \frac{1}{2}\left[\frac{\varepsilon}{1-\alpha}\widetilde{\sigma}_\pi(\hat{z}_t)\right]^2 && \approx \mathcal{V}_{0,4} + \mathcal{V}_{s,4}\hat{s}_t + \mathcal{V}_{\nu,4}\hat{\nu}_t\\
\mathcal{V}_t\left(e^{\frac{\varepsilon}{1-\alpha+\alpha\varepsilon}w_{3,t+1}}\right) &= \frac{1}{2}\left[\frac{\varepsilon}{1-\alpha}\frac{\eta\widetilde{\Pi}^{\varepsilon-1}}{1-\eta\widetilde{\Pi}^{\varepsilon-1}}\widetilde{\sigma}_\pi(\hat{z}_t)\right]^2 && \approx \mathcal{V}_{0,5} + \mathcal{V}_{s,5}\hat{s}_t + \mathcal{V}_{\nu,5}\hat{\nu}_t
\end{aligned}
$$

with $\widetilde{\sigma}_\ell(\hat{z}_t) \doteq \left[-\psi_{\ell u}\phi + \psi_{\ell s}\Lambda(\hat{s}_t)\widetilde{\sigma}_c(\hat{z}_t) + \psi_{\ell\nu}\Upsilon(\nu_t)\widetilde{\sigma}_\pi(\hat{z}_t)\right]\sigma$.

Step 3. Linearize and match coefficients or, equivalently, solve matrix equation (5).

6.2.6. Quality of the approximation

We calibrate the parameters with the values in table 3. The level of the interest rate in the risky steady state $i^* = -\ln\beta + \gamma\mu - \mathcal{V}_{0,1}$ and the employment subsidy τ are chosen by the government to correct the risky steady state to a model without monopolistic competition and zero steady-state inflation. Moreover, we calibrate ξ_1 and ξ_2 so the approximate elasticities $\frac{\partial\pi_t}{\partial\hat{s}_t}$ and $\frac{\partial\widetilde{c}_t}{\partial\hat{s}_t}$ are zero (see Lopez et al., 2015). Accordingly, the homoskedasticity of technology implies the homoskedasticity of inflation and consumption.

Finally, cashflows can be written in form (14) while market dividends approximate as:

$$
d_t = \ln(\alpha) + c_t - \frac{\alpha}{1-\alpha}\hat{mc}_t = \ln(\alpha) + a_t - \frac{\gamma(1-\alpha)+\widetilde{\varphi}}{\alpha}(\widetilde{c}_t - \widetilde{c}) + \frac{\gamma(1-\alpha)}{\alpha}\xi_2\hat{s}_t
$$

Figure 8 compares the global solution with our generalized affine approximation method and with the benchmark approximations. The plots show an overall good performance of our affine

	Parameter		Value
New Keynesian block	γ	Utility curvature in market and home consumption	2
	$1/\varphi$	Quasi-Frisch's labor supply elasticity	1
	β	Subjective discount factor	.9945
	$1-\alpha$	Labor share in value added	2/3
	ε	Elasticity of substitution in Dixit-Stiglitz aggregator	2
	$1/(1-\eta)$	Average price duration (in months)	8
	ϕ_π	Policy response coefficient to inflation movements	1.23
Habit block	ξ_1	Financial spillover onto the intertemporal rate of substitution	−.0002
	ξ_2	Financial spillover onto the intratemporal rate of substitution	−.0110
	ρ_s	Habit persistence	.9918
Exogenous block	μ	Mean technology growth	.0032
	ρ_u	Persistence of the conditional mean of technology growth	.8014
	σ	Conditional volatility of technology	.0163
	ϕ	Relative volatility of the conditional mean of technology	.1242

Table 3: Deep parameters and their calibration (monthly frequency) in the model by Lopez et al. (2015).

approximation, which captures well the level, amplitude and shape of the term structures of risk premia and volatilities, including the initially downward-sloping term structure of market equity. The linear perturbation approximation around the risky steady state displays the usual bias in representing long-run pricing properties due to the small radius of convergence of map Λ. The alternative affine approximation is severely inaccurate by its inability to capture a dynamic risk adjustment due to the interaction between the time-varying function Λ and endogenous innovations to consumption. The standard affine method has therefore a hard time in accounting for a precautionary savings effect. This distortion is particularly severe for the term structure of nominal yields, as the spillover of surplus consumption on inflation has a dramatic effect on its volatility.

Figure 4 shows multiperiod Euler equation errors. The accuracy of our global solution in terms of conventional 1-step ahead Euler equation errors is consistently lower than -3 and is comparable to values typically retained in the extant literature, and remains with maxima of around -2 over arbitrarily long horizons. The risk-adjusted loglinearized solution for quantities that forms the basis of the generalized affine approximation also shows relatively small errors; notably, the lower accuracy of the loglinearized solution does not translate in substantially larger Euler equation errors over long horizons when contrasted with the projected solution.

6.3. Application 4: Gourio (2012)

A representative consumer has recursive preferences in consumption (C_t) and leisure (L_t) with elasticity of intertemporal substitution ρ, risk aversion coefficient γ and rate of time preference β:

$$v_t = c_t + \chi l_t + \frac{1}{1-\rho} \ln\left(1 - \beta + \beta e^{(1-\rho)(x_t - c_t - \chi l_t)}\right), \quad x_t \doteq \frac{1}{1-\gamma} \ln E_t e^{(1-\gamma)v_{t+1}}$$

She lives in a production economy and chooses output $Y_t = (A_t N_t)^{1-\alpha} K_t^\alpha$ and the trajectories of labor hours $N_t = 1 - L_t$ and capital K_t, whose accumulation is subject to adjustment costs:

$$K_{t+1} = e^{\xi^P_{t+1}} \left[1 - \delta + \Phi\left(\frac{I_t}{K_t}\right)\right] K_t = e^{\xi^P_{t+1}} \left[\frac{I_t}{K_t} - \frac{\eta}{2}\left(\frac{I_t}{K_t} - \delta - e^\mu + 1\right)^2\right] K_t$$

Output is devoted to consumption or to investment, $Y_t = C_t + I_t$. Log technology $a_t = a_t^P + a_t^T$ consists of permanent and transitory components driven by a normal innovation and a jump component that represents rare disaster events:

$$\begin{aligned}
\Delta a^P_{t+1} &= \mu + \sigma_a \varepsilon^a_{t+1} + \xi^P_{t+1} & a^T_{t+1} &= \rho_a a^T_t + \xi^T_{t+1} - \xi^P_{t+1} \\
\xi^P_{t+1} &= (\mu_P - \frac{1}{2}\sigma_P^2) j_{t+1} + \sigma_P \sqrt{j_{t+1}} \varepsilon^P_{t+1}, & \xi^T_{t+1} &= (\mu_T - \frac{1}{2}\sigma_T^2) j_{t+1} + \sigma_T \sqrt{j_{t+1}} \varepsilon^T_{t+1} \\
j_{t+1} &\sim B(p_t) & \ln(p_{t+1}) &= (1 - \rho_p)\ln(p) + \rho_p \ln(p_t) + \sigma_p \varepsilon^p_{t+1}
\end{aligned}$$

and disasters are independent of $[\varepsilon_t^a; \varepsilon_t^p; \varepsilon_t^P; \varepsilon_t^T] \sim Niid(0, I_4)$. Let $xa_t \doteq x_t - a_t^P$ denote the detrended version of log variable x.

Joint optimality of consumption, labor, investment and capital accumulation imply, in form (1):

$$0 = \ln E_t e^{f(y_t, z_t, y_{t+1}, z_{t+1})}$$

$$f(y_t, z_t, y_{t+1}, z_{t+1}) = \begin{bmatrix}
\ln(\beta) - \rho(\mu + \sigma_a \varepsilon^a_{t+1}) + (1-\gamma)\xi^P_{t+1} - \rho \Delta ca_{t+1} + \chi(1-\rho)\Delta l_{t+1} + (\rho - \gamma)(va_{t+1} + \sigma_a \varepsilon^a_{t+1} - xa_t) + w^i_t + w^k_{t+1} \\
(1-\rho)(va_t - ca_t - \chi l_t) - w^v_t \\
(1-\gamma)(va_{t+1} + \sigma_a \varepsilon^a_{t+1} + \xi^P_{t+1} - xa_t) \\
l_t - \ln(1 - e^{n_t}) \\
ca_t - l_t - \ln\left(\frac{1-\alpha}{\chi}\right) - (1-\alpha)a^T_t + \alpha(n_t - ka_t) \\
\ln(e^{ck_t} + e^{ik_t}) - (1-\alpha)(a^T_t + n_t - ka_t) \\
w^i_t - \ln[1 - \eta(e^{ik_t} - \delta - e^\mu + 1)] \\
w^k_t - \ln\left(\alpha e^{ck_t} - (1-\alpha)e^{ik_t} + \frac{1 - \delta + e^{ik_t} - \frac{\eta}{2}(e^{ik_t} - \delta - e^\mu + 1)^2}{1 - \eta(e^{ik_t} - \delta - e^\mu + 1)}\right) \\
w^v_t - \ln\left(1 + e^{\ln(\beta) + (1-\rho)(\mu + xa_t - ca_t - \chi l_t)}\right)
\end{bmatrix}$$

for the vector of decision variables $y_t = [ca_t; l_t; xa_t; n_t; ia_t; va_t; w^i_t; w^k_t; w^v_t]$ and state vector $z_t = [ka_t; \lambda_t - \lambda; a^T_t; \varepsilon^a_t; \varepsilon^{\xi P}_t]$:

$$\begin{bmatrix} ka_{t+1} + \mu \\ \lambda_{t+1} - \lambda \\ a^T_{t+1} \\ \varepsilon^a_{t+1} \\ \varepsilon^P_{t+1} \end{bmatrix} = \begin{bmatrix} ka_t + \ln[1 - \delta + e^{ik_t} - \frac{\eta}{2}(e^{ik_t} - \delta - e^\mu + 1)^2] \\ \rho_p(\lambda_t - \lambda) \\ \rho_a a^T_t + [\mu_T - \mu_P - \frac{1}{2}(\sigma_T^2 - \sigma_P^2)]e^{\lambda_t} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\sigma_a & 0 & 0 & 0 \\ 0 & \sigma_p & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon^a_{t+1} \\ \varepsilon^p_{t+1} \\ \varepsilon^{\xi T}_{t+1} \\ \varepsilon^{\xi P}_{t+1} \end{bmatrix}$$

where $\lambda_t \doteq \ln(p_t)$, $\varepsilon^{\xi T}_{t+1} \doteq \xi^T_{t+1} - E_t \xi^T_{t+1}$ and $\varepsilon^{\xi P}_{t+1} \doteq \xi^P_{t+1} - E_t \xi^P_{t+1}$.

6.3.1. Affine approximation

We apply the algorithm in section 2.

Parameter	value
Capital share, α	0.34
Capital depreciation rate, δ	0.02
Leisure preference parameter, χ	2.33
Discount factor, β	0.99
Capital adjustment cost, η	1.7
Elasticity of substitution, $\frac{1}{1-(1+\chi)(1-\rho)}$	2
Risk aversion parameter, γ	10
Mean TFP growth (in %), μ	0.5
Standard deviation of TFP growth (in %), σ_a	1
Persistence of TFP, ρ_a	0.71
Mean permanent shock, μ_P	-0.007
Standard deviation of permanent shock, σ_P	0.092
Mean transitory shock, μ_P	-0.055
Standard deviation of transitory shock (in %), σ_P	0.041
Average probability of disaster (in %), p	1.00
Standard deviation of shock to disaster probability, σ_P	0.44
Persistence of disaster risk, ρ_p	0.90

Table 4: Deep parameters and their calibration (quarterly frequency) in the model by Gourio (2012).

Step 1. Write expectational equations in terms of a certainty-equivalent and an entropy terms:

$$0 = \ln(\beta) - \rho\mu + (1-\gamma)E_t\xi^P_{t+1} - \rho E_t \Delta ca_{t+1} + \chi(1-\rho)E_t\Delta l_{t+1} + (\rho-\gamma)(E_t va_{t+1} - xa_t) + w^i_t + E_t w^k_{t+1}$$
$$+ \mathcal{V}_t\left(e^{-\rho\sigma_a\varepsilon^a_{t+1}+(1-\gamma)\xi^P_{t+1}-\rho ca_{t+1}+\chi(1-\rho)l_{t+1}+(\rho-\gamma)(va_{t+1}+\sigma_a\varepsilon^a_{t+1})+w^k_{t+1}}\right)$$
$$0 = (1-\gamma)(E_t va_{t+1} + E_t\xi^P_{t+1} - xa_t) + \mathcal{V}_t\left(e^{(1-\gamma)(va_{t+1}+\sigma_a\varepsilon^a_{t+1}+\xi^P_{t+1})}\right)$$

Step 2. Conjecture an approximate affine solution for decision variables in the state vector and characterize the entropy terms by noting that the ccgf of exogenous innovations in state variables has shape:

$$\mathcal{V}_t\left(e^{u_a\varepsilon^a_{t+1}+u_p\varepsilon^p_{t+1}+u_T\xi^T_{t+1}+u_P\xi^P_{t+1}}\right) = \frac{u_a^2}{2} + \frac{u_p^2}{2} + \ln\left[1 + e^{\lambda_t}\left(e^{u_T(\mu_T - \frac{1-u_T}{2}\sigma_T^2)+u_P(\mu_P - \frac{1-u_P}{2}\sigma_P^2)} - 1\right)\right] - u_T E_t\xi^T_{t+1} - u_P E_t\xi^P_{t+1}$$

Step 3. Linearize and match coefficients or, equivalently, solve matrix equation (5).

6.3.2. Quality of the approximation

We calibrate the parameters with the values in table 4. Cashflows can be written in form (14). We consider real bonds with log cashflow process $d_t = 0$, consumption claims with $d_t = c_t$, and market dividend claims with $d_t = 2c_t$.

Figures 5 and 9 compare the global solution with our affine approximation method. In this context in which maps λ and σ are constant—using the notation in representation (1)—the entropy-based affine approximation is equivalent to a linear perturbation around the risky steady state. This example is particularly challenging because capital is closer to linear in levels rather than in logs and the dynamic risk correction is closer to linear in the disaster probability rather than in its log.

The plots show an overall reasonably good performance of our entropy-based approximation in capturing the level, amplitude and shape of the term structures of risk premia and return volatilities. Moreover, our entropy-based affine solution associates with relatively small Euler equation errors; notably, the lower accuracy of the loglinearized solution does not translate in substantially larger multiperiod Euler equation errors when contrasted with the projected solution.

7. Conclusion

Exponential-affine approximations have routinely been used in the finance literature as ad-hoc approximation strategies. As we show, the desirable implications of these methods find a formal justification in the theory of local perturbation approximations based on the implicit function and Taylor theorems. Our entropy-based approximation generalizes substantially the applicability of the methodology while achieving a tight relation with perturbations around a stochastic stationary point. The resulting approximation technique offers explicit formulas and numerical routines to approximate equilibrium quantities and asset prices in a large class of dynamic macro-finance models as well as conditions for existence and uniqueness of the approximate local dynamics.

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Appendix

A. Proof of proposition 1.a

We follow Klein (2000) and consider the generalized Schur factorization of Γ and Ξ^q, with unitary $Q, Z \in \mathbb{C}^{n_y+n_z \times n_y+n_z}$ and upper triangular matrices $S, T \in \mathbb{C}^{n_y+n_z \times n_y+n_z}$ such that:

$$Q\Gamma Z = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \qquad Q\Xi^q Z = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \qquad Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},\ Z^* = \begin{bmatrix} Z_{11}^* & Z_{21}^* \\ Z_{12}^* & Z_{22}^* \end{bmatrix}$$

with Z^* the conjugate transpose of Z, where $S_{11}, T_{11} \in \mathbb{C}^{n_z \times n_z}$, $S_{22}, T_{22} \in \mathbb{C}^{n_y \times n_y}$, $Z_{11} \in \mathbb{C}^{n_z \times n_z}$, $Z_{12} \in \mathbb{C}^{n_z \times n_y}$, and matrices S, T are sorted with generalized eigenvalues $\alpha(\Gamma, \Xi) = \{t_{ii}/s_{ii}, i = 1, ..., n_y + n_z\}$ in increasing order as $|t_{ii}/s_{ii}| < 1, i = 1, ..., n_z$ and $|t_{ii}/s_{ii}| > 1, i = n_z + 1, ..., n_z + n_y$. The dependence of Q, S, T, Z on q is not denoted explicitly for simplicity.

We rewrite the matrix equation that describes the affine solution (12) as:

$$\Gamma \begin{bmatrix} I_{n_z} \\ \Psi^q \end{bmatrix} [g(\overline{y}^q, \overline{z}^q)\Psi^q + g_2(\overline{y}^q, \overline{z}^q)](z_t^q - \overline{z}^q) = \Xi \begin{bmatrix} I_{n_z} \\ \Psi^q \end{bmatrix} (z_t^q - \overline{z}^q)$$

$$\text{or:} \quad Q\Gamma ZZ^* \begin{bmatrix} I_{n_z} \\ \Psi^q \end{bmatrix} E_t(z_{t+1}^q - \overline{z}^q) = Q\Xi ZZ^* \begin{bmatrix} I_{n_z} \\ \Psi^q \end{bmatrix} (z_t^q - \overline{z}^q) \quad \Leftrightarrow \quad S\,E_t \begin{bmatrix} x_{z,t+1} \\ x_{y,t+1} \end{bmatrix} = T \begin{bmatrix} x_{z,t} \\ x_{y,t} \end{bmatrix} \qquad \text{(A.1)}$$

with

$$\begin{bmatrix} x_{z,t} \\ x_{y,t} \end{bmatrix} \doteq Z^* \begin{bmatrix} I_{n_z} \\ \Psi^q \end{bmatrix} (z_t^q - \overline{z}^q), \qquad x_{z,t} \in \mathbb{R}_t^{n_z},\ x_{y,t} \in \mathbb{R}_t^{n_y} \qquad \text{(A.2)}$$

Note that the upper triangular matrices S_{11} and T_{22} are invertible, as their respective eigenvalues $\{s_{ii}, i = 1, ..., n_z\}$ and $\{t_{ii}, i = n_z + 1, ..., n_z + n_y\}$ are nonzero by the conditions for saddle-path stability.

By the stability requirement, $\lim |E_t z_{t+N}^q| < \infty$, equation (A.1) implies:

$$x_{y,t} = T_{22}^{-1} S_{22} E_t x_{y,t+1} = (T_{22}^{-1} S_{22})^N E_t x_{y,t+N} \overset{N\to\infty}{\to} 0$$

as the eigenvalues of the upper triangular matrix $T_{22}^{-1} S_{22}$ coincide with $\{s_{ii}/t_{ii}, i = n_z + 1, ..., n_z + n_y\}$, hence lie within the unit circle. Using definition (A.2), it follows that $\Psi^q = -(Z_{22}^*)^{-1} Z_{12}^* = Z_{21} Z_{11}^{-1}$, where the last equality and invertibility owe to the orthonormality of matrix Z. Orthonormality of Z also implies $Z_{11}^* - Z_{21}^* (Z_{22}^*)^{-1} Z_{12}^* = Z_{11}^{-1}$. Therefore, equation (A.1) implies:

$$E_t x_{z,t+1} = S_{11}^{-1} T_{11} x_{z,t}, \quad x_{z,t} = (Z_{11}^* + Z_{21}^* \Psi^q)(z_t^q - \overline{z}^q) = Z_{11}^{-1}(z_t^q - \overline{z}^q)$$

hence $E_t(z_{t+1}^q - \overline{z}^q) = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1} (z_t^q - \overline{z}^q)$, so the spectrum of matrix $g_1(\overline{y}^q, 0)\Psi^q + g_2(\overline{y}^q, 0)$ is:

$$\left\{\lambda \in \mathbb{C} : \det[Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1} - \lambda I_{n_z}] = 0\right\} = \left\{\frac{t_{ii}}{s_{ii}}, i = 1, ..., n_z\right\}$$

Since the spectral radius of the matrix is less than unity, the state vector has stable dynamics.

B. Proof of proposition 1.b

We are looking for functionals $y_t^q = y(z_t^q, q, \tau)$ and $z_{t+1}^q = z(z_t^q, q, \varepsilon_{t+1}, \tau)$, and we rewrite the dynamic system in functional form as:

$$F([y^q, z^q], \varepsilon, q, \tau) = \left\{ \begin{bmatrix} h(y_t^q, z_t^q) + f_3 E_t y_{t+1}^q + f_4 E_t z_{t+1}^q + \tau w(z_t^q, q, \tau) + (1-\tau)\widetilde{\mathcal{V}}(z_t^q, q) \\ z_{t+1}^q - g(y_t^q, z_t^q) - \lambda(z_t^q)(y_{t+1}^q - E_t y_{t+1}^q) - q\sigma(z_t^q)\varepsilon_{t+1} \end{bmatrix} \right\}_{t=0}^{\infty}$$

whose tth coordinate maps $q \in [0,1]$, $\tau \in [0,1]$, $\varepsilon_{t+1} \in \mathbb{R}^{n_\varepsilon}$ and essentially-bounded functions $[y_t^q; z_t^q; y_{t+1}^q; z_{t+1}^q]$ of the history of shocks $\{\varepsilon_s\}_{s \le t+1}$ into the Banach space of essentially-bounded functions of the history of shocks $\{\varepsilon_s\}_{s \le t}$. The derivative operator of the tth coordinate of map F evaluated at the expansion point is:

$$D_{F,t}[\hat{y}^q; \hat{z}^q] = \Gamma \begin{bmatrix} E_t \hat{z}_{t+1}^q \\ E_t \hat{y}_{t+1}^q \end{bmatrix} - \Xi \begin{bmatrix} \hat{z}_t^q \\ \hat{y}_t^q \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} f_4 & f_3 \\ I_{n_z} & 0 \end{bmatrix}, \quad \Xi = \begin{bmatrix} -h_2(\bar{y}^q, \bar{z}^q) - \widetilde{\mathcal{V}}_1(\bar{z}^q, q) & -h_1(\bar{y}^q, \bar{z}^q) \\ g_2(\bar{y}^q, \bar{z}^q) & g_1(\bar{y}^q, \bar{z}^q) \end{bmatrix}$$

and it maps an a.s.-bounded sequence of perturbed arguments $\{\hat{y}_t^q; \hat{z}_t^q\}_{t=0}^{\infty}$ into a unique a.s.-bounded process $u^q = \{u_t^q\}_{t=0}^{\infty}$ that is a measurable function of the history of shocks. Note how the derivative operator is well-defined because the ccgf of exogenous shocks exists and is differentiable.[17]

Risky steady state is a saddle point ⇒ Locally unique and differentiable implicit functions. The goal is to show that maps $[y^q, z^q]$ of $[\varepsilon, q, \tau]$ are defined uniquely and are differentiable on a sufficiently small neighborhood of $[0, q, 0]$. If we can invoke the implicit function theorem, the proof follows immediately. To be able to invoke the implicit function theorem in Banach spaces (e.g. Lang, 1993, 364), we have to prove that the derivative operator around the expansion point is invertible as a continuous (and hence bounded) linear operator.[18]

In particular, an a.s.-bounded process $\{u_t^q\}_{t=0}^{\infty}$ maps into unique a.s.-bounded processes $\{\hat{y}_t^q; \hat{z}_t^q\}_{t=0}^{\infty}$ if and only if the expansion point is a saddle point. To prove this claim, we write the derivative as:

$$Q\, D_{F,t}[\hat{y}^q; \hat{z}^q] = S \begin{bmatrix} E_t x_{z,t+1} \\ E_t x_{y,t+1} \end{bmatrix} - T \begin{bmatrix} x_{z,t} \\ x_{y,t} \end{bmatrix}, \text{ with } \begin{bmatrix} x_{z,t} \\ x_{y,t} \end{bmatrix} \doteq Z^* \begin{bmatrix} \hat{z}_t^q \\ \hat{y}_t^q \end{bmatrix}$$

where Q, S, T, Z constitute the Schur factorization of Γ and Ξ. The dependence of Q, S, T, Z on q is not denoted explicitly for simplicity. We then note that the derivative operator in equation:

$$D_F[\hat{y}^q; \hat{z}^q] = u^q \quad \Leftrightarrow \quad S \begin{bmatrix} E_t x_{z,t+1} \\ E_t x_{y,t+1} \end{bmatrix} = T \begin{bmatrix} x_{z,t} \\ x_{y,t} \end{bmatrix} + v_t^q, \quad \begin{bmatrix} v_{z,t}^q \\ v_{y,t}^q \end{bmatrix} \doteq Q u_t^q$$

[17] Also, the existence of the ccgf of exogenous shocks is not a local property and yet is a necessary regularity condition, as the moment $E_t e^{\alpha(z_t) q \varepsilon_{t+1}}$ for a real map α needs not exist otherwise even for arbitrarily small $q > 0$. Jin and Judd (2002) and Kim et al. (2008) make a similar point about the existence of moments of shocks.

[18] We also require $[y^q, z^q]$ to be in an open set of the topology of a.s.-bounded functions. As in the case of linear perturbations around the deterministic steady state, we can guarantee this property in the topology of essentially-bounded functions if exogenous shocks have a.s.-bounded support (Jin and Judd, 2002). Note that the reasoning is local; in particular, for a z_t^q in a neighborhood of $\bar{z}^q$ we have that z_{t+1}^q is in the same neighborhood only under a sufficiently small $q > 0$. Whether $q = 1$ is sufficiently small will in turn depend on whether $\sigma(\bar{z}^q)$ is and will be a practical question about the quality of the approximation.

can be inverted as:

$$
\begin{aligned}
x_{y,t} &= T_{22}^{-1} S_{22} E_t x_{y,t+1} - T_{22}^{-1} v_{y,t}^{q} \overset{N\to\infty}{\to} - \sum_{j=0}^{\infty} (T_{22}^{-1} S_{22})^j T_{22}^{-1} E_t v_{y,t+j}^{q} \\
E_t x_{z,t+1} &= S_{11}^{-1} T_{11} x_{z,t} + S_{11}^{-1} (T_{12} x_{y,t} - S_{12} E_t x_{y,t+1}) + S_{11}^{-1} v_{z,t}^{q}
\end{aligned}
$$

if and only if T_{22} and S_{11} are invertible and $T_{22}^{-1} S_{22}$ and $S_{11}^{-1} T_{11}$ have eigenvalues inside the unit circle; this is the definition of saddle-point stability of the risky steady state of system (13). Orthonormal matrices Q and Z map v^q and $[x_z; x_y]$ back into the original variables u^q and $[y^q; z^q]$.

The invertibility of the derivative operator evaluated at the expansion point implies that we can rely on the implicit function theorem to characterize the functions of the history of shocks with the target form $y_t^q = y(z_t^q, q, \tau)$ and $z_{t+1}^q = z(z_t^q, q, \varepsilon_{t+1}, \tau)$ that solve $F([y^q, z^q], \varepsilon, q, \tau) = 0$. Namely, these functions are unique and differentiable in a neighborhood of the expansion point.

Locally unique and differentiable implicit functions ⇒ Coefficients from first-order Taylor approximation equal coefficients from affine approximation. We can now approximate the local solution around the point $[z_t^q, \tau] = [\bar{z}^q, 0]$ via Taylor theorem. Note how no expansion in q will take place. We are looking to identify the approximate functions:

$$
\begin{aligned}
y_t^q &= y(\bar{z}^q, q, 0) + y_1(\bar{z}^q, q, 0)(z_t^q - \bar{z}^q) + y_3(\bar{z}^q, q, 0)\tau \\
z_{t+1}^q &= z(\bar{z}^q, q, \varepsilon_{t+1}, 0) + z_1(\bar{z}^q, q, \varepsilon_{t+1}, 0)(z_t^q - \bar{z}^q) + z_4(\bar{z}^q, q, \varepsilon_{t+1}, 0)\tau \\
x_{t+1}^q &= x(\bar{z}^q, q, \varepsilon_{t+1}, 0) + x_1(\bar{z}^q, q, \varepsilon_{t+1}, 0)(z_t^q - \bar{z}^q) + x_4(\bar{z}^q, q, \varepsilon_{t+1}, 0)\tau
\end{aligned}
$$

It is useful to define the derivative of a differentiable matrix $\lambda : \mathbb{R}^{n_z} \to \mathbb{R}^{n_y \times n_x}$ as:

$$
\lambda_1(0) = \begin{bmatrix} \frac{\partial \lambda_{(1,:)}(0)}{\partial \hat{z}_t} \\ \vdots \\ \frac{\partial \lambda_{(n_y,:)}(0)}{\partial \hat{z}_t} \end{bmatrix} \in \mathbb{R}^{n_z n_y \times n_x}, \quad \frac{\partial \lambda_{(i,:)}(0)}{\partial \hat{z}_t} \doteq \begin{bmatrix} \frac{\partial \lambda_{(i,1)}(0)}{\partial \hat{z}_{1,t}} & \cdots & \frac{\partial \lambda_{(i,n_x)}(0)}{\partial \hat{z}_{1,t}} \\ \vdots & & \vdots \\ \frac{\partial \lambda_{(i,1)}(0)}{\partial \hat{z}_{n_z,t}} & \cdots & \frac{\partial \lambda_{(i,n_x)}(0)}{\partial \hat{z}_{n_z,t}} \end{bmatrix} \in \mathbb{R}^{n_z \times n_x}
$$

for each row $i = 1, ..., n_z$.

A Taylor approximation of the equilibrium conditions around point $[z_t^q, \tau] = [\bar{z}^q, 0]$ yields:

$$
\begin{aligned}
z(\bar{z}^q, q, \varepsilon_{t+1}, 0) &= g[y(\bar{z}^q, q, 0), \bar{z}^q] + \lambda(\bar{z}^q)(E_{t+1} - E_t) y[z(\bar{z}^q, q, \varepsilon_{t+1}, 0), q, 0] + q\sigma(\bar{z}^q)\varepsilon_{t+1} \\
&= g[y(\bar{z}^q, q, 0), \bar{z}^q] + \sigma_z(\bar{z}^q, q)\varepsilon_{t+1}, \quad \sigma_z(z_t^q, q) \doteq q[I_{n_z} - \lambda(z_t^q) y_1(\bar{z}^q, q, 0)]^{-1} \sigma(z_t^q) \\
z_1(\bar{z}^q, q, \varepsilon_{t+1}, 0)(z_t^q - \bar{z}^q) &= [g_1^q y_1(\bar{z}^q, q, 0) + g_2^q](z_t^q - \bar{z}^q) + [I_{n_z} \otimes (z_t^q - \bar{z}^q)'] \lambda_1(\bar{z}^q)(E_{t+1} - E_t) y[z(\bar{z}^q, q, \varepsilon_{t+1}, 0), q, 0] \\
&\quad + \lambda(\bar{z}^q)(E_{t+1} - E_t) y_1[z(\bar{z}^q, q, \varepsilon_{t+1}, 0), q, 0] z_1(\bar{z}^q, q, \varepsilon_{t+1}, 0)(z_t^q - \bar{z}^q) \\
&\quad + [I_{n_z} \otimes (z_t^q - \bar{z}^q)'] \sigma_1(\bar{z}^q) q \varepsilon_{t+1} \\
&= [g_1^q y_1(\bar{z}^q, q, 0) + g_2^q](z_t^q - \bar{z}^q) + [I_{n_z} \otimes (z_t^q - \bar{z}^q)'] \sigma_{1,z}(\bar{z}^q, q)\varepsilon_{t+1} \\
z_4(\bar{z}^q, q, \varepsilon_{t+1}, 0) &= g_1^q y_3(\bar{z}^q, q, 0) + [I_{n_z} - \lambda(z_t^q) y_1(\bar{z}^q, q, 0)]^{-1} \lambda(\bar{z}^q)(E_{t+1} - E_t) y_3(\bar{z}^q, q, 0) \\
&= g_1^q y_3(\bar{z}^q, q, 0)
\end{aligned}
$$

where $g_1^q \doteq g_1[y(\bar{z}^q, q, 0), \bar{z}^q]$ and $g_2^q \doteq g_2[y(\bar{z}^q, q, 0), \bar{z}^q]$, with the auxiliary variable:

$$\begin{aligned}
x(\bar{z}^q, q, \varepsilon_{t+1}, 0) &= h[y(\bar{z}^q, q, 0), \bar{z}^q] + f_3 y[z(\bar{z}^q, q, \varepsilon_{t+1}, 0), q, 0] + f_4 z(\bar{z}^q, q, \varepsilon_{t+1}, 0) \\
&= h[y(\bar{z}^q, q, 0), \bar{z}^q] + f_3 y(\bar{z}^q, q, 0) + f_4 \bar{z}^q + [f_3 y_1(\bar{z}^q, q, 0) + f_4]\sigma_z(\bar{z}^q, q)\varepsilon_{t+1} \\
x_1(\bar{z}^q, q, \varepsilon_{t+1}, 0)(z_t^q - \bar{z}^q) &= \left[f_1^q y_1(\bar{z}^q, q, 0) + f_2^q + f_3 y_1[z(\bar{z}^q, q, \varepsilon_{t+1}, 0), q, 0] z_1(\bar{z}^q, q, \varepsilon_{t+1}, 0) + f_4 z_1(\bar{z}^q, q, \varepsilon_{t+1}, 0)\right](z_t^q - \bar{z}^q) \\
&= \left[f_1^q y_1(\bar{z}^q, q, 0) + f_2^q + [f_3 y_1(\bar{z}^q, q, 0) + f_4]\left(g_1^q y_1(\bar{z}^q, q, 0) + g_2^q\right)\right](z_t^q - \bar{z}^q) \\
&\quad + [f_3 y_1(\bar{z}^q, q, 0) + f_4][I_{n_z} \otimes (z_t^q - \bar{z}^q)']\sigma_{1,z}(\bar{z}^q, q)\varepsilon_{t+1} \\
x_4(\bar{z}^q, q, \varepsilon_{t+1}, 0) &= f_1^q y_3(\bar{z}^q, q, 0) + f_3 y_3[z(\bar{z}^q, q, \varepsilon_{t+1}, 0), q, 0] + [f_3 y_1[z(\bar{z}^q, q, \varepsilon_{t+1}, 0), q, 0] + f_4]\, z_4(\bar{z}^q, q, \varepsilon_{t+1}, 0) \\
&= \left[f_1^q + f_3 + [f_3 y_1(\bar{z}^q, q, 0) + f_4] g_1^q\right] y_3(\bar{z}^q, q, 0)
\end{aligned}$$

where $f_1^q \doteq f_1[y(\bar{z}^q, q, 0), \bar{z}^q]$ and $f_2^q \doteq f_2[y(\bar{z}^q, q, 0), \bar{z}^q]$. In the derivation we used the property of the approximate solution:

$$\begin{aligned}
y[z(\bar{z}^q, q, \varepsilon_{t+1}, 0), q, 0] &= y(\bar{z}^q, q, 0) + y_1(\bar{z}^q, q, 0)\sigma_z(\bar{z}^q, q)\varepsilon_{t+1} \\
y_1[z(\bar{z}^q, q, \varepsilon_{t+1}, 0), q, 0] z_1(\bar{z}^q, q, \varepsilon_{t+1}, 0) &= y_1(\bar{z}^q, q, 0) z_1(\bar{z}^q, q, \varepsilon_{t+1}, 0)
\end{aligned}$$

that follows from $y[z(z_t^q, q, \varepsilon_{t+1}, 0), q, 0] = y(\bar{z}^q, q, 0) + y_1(\bar{z}^q, q, 0)[z(z_t^q, q, \varepsilon_{t+1}, 0) - \bar{z}^q]$.

Note that approximate entropy is:

$$w(\bar{z}^q, q, 0) = \mathcal{V}\left[e^{x(\bar{z}^q, q, \varepsilon_{t+1}) + x_1(\bar{z}^q, q, \varepsilon_{t+1})(z_t^q - \bar{z}^q)} | \bar{z}^q\right] = \kappa\left[(f_3 y_1(\bar{z}^q, q, 0) + f_4)\sigma_z(\bar{z}^q, q); \bar{z}^q\right] \tag{B.3}$$

hence we identify $[y(\bar{z}^q, q, 0), y_1(\bar{z}^q, q, 0)]$ using equation $E_t x_{t+1}^q + \tau w(z_t^q, q, \tau) + (1 - \tau)\widetilde{\mathcal{V}}(z_t^q, q) = 0$ and matching coefficients as:

$$0 = h[y(\bar{z}^q, q, 0), \bar{z}^q] + f_3 y(\bar{z}^q, q, 0) + f_4 \bar{z}^q + \widetilde{\mathcal{V}}(\bar{z}^q, q) \tag{B.4}$$

$$0 = f_1^q y_1(\bar{z}^q, q, 0) + f_2^q + [f_3 y_1(\bar{z}^q, q, 0) + f_4][g_1^q y_1(\bar{z}^q, q, 0) + g_2^q] + \widetilde{\mathcal{V}}_1(\bar{z}^q, q) \tag{B.5}$$

$$0 = \left[f_1^q + f_3 + [f_3 y_1(\bar{z}^q, q, 0) + f_4] g_1^q\right] y_3(\bar{z}^q, q, 0) + w(\bar{z}^q, q^0) - \widetilde{\mathcal{V}}(\bar{z}^q, q) \tag{B.6}$$

Matrix equations (B.4) and (B.5) coincide with matrix equation (12). It follows that $\widetilde{z}^q = \bar{z}^q$, $\widetilde{y}^q = y(\bar{z}^q, 1, 0)$ and $\Psi^q = y_1(\bar{z}^q, 1, 0)$. Therefore, matrix equations (B.4) and (B.5) at $q = 1$ coincide with matrix equation (5); affine coefficients can be interpreted as the coefficients from a first-order perturbation around the risky steady state $(y_t^q, z_t^q) = (\bar{y}^q, \bar{z}^q)$ evaluated at $q \in [0, 1]$ and $\varepsilon_{t+1} = 0$.

Finally, $\widetilde{z}^q = \bar{z}^q$ and $\Psi^q = y_1(\bar{z}^q, 1, 0)$ imply $w(\bar{z}^q, q, 0) = \widetilde{\mathcal{V}}(\bar{z}^q, q)$ by equation (B.3). It follows that $y_3(\bar{z}^q, q, 0) = 0$ by equation (B.6). The local *slope* of the solution with respect to τ is zero, a property that needs not of course extend to higher-order curvature terms.

C. *Proof of proposition 3*

We conjecture that the price-dividend ratio of the n-period ahead cashflow strip has the exponential-affine shape $P_{d,t}^{(n)}/D_t = e^{A^{(n)} + B_z^{(n)} \hat{z}_t}$, and use the no-arbitrage relation:

$$P_{d,t}^{(n)} = E_t^{\mathbb{P}}[e^{m_{t+1}} P_{dt+1}^{(n-1)}], \qquad P_{d,t}^{(0)} = D_t$$

to verify the conjecture as:

$$
\begin{aligned}
e^{A^{(n)}+B_z^{(n)}\hat{z}_t} &= E_t^{\mathbb{P}}[e^{m_{t+1}+A^{(n-1)}+B_z^{(n-1)}\hat{z}_{t+1}+\Delta d_{t+1}}] \\
&= e^{-r(\hat{z}_t)+\kappa[-\gamma(\hat{z}_t)';z_t]+A^{(n-1)}+B_z^{(n-1)}A\hat{z}_t+\mu_d+C\hat{z}_t+\mathcal{V}_t[\exp(-\gamma(\hat{z}_t)'\varepsilon_{t+1}+B_z^{(n-1)}B(\hat{z}_t)\varepsilon_{t+1}+D(\hat{z}_t)\varepsilon_{t+1})]} \\
&= e^{-r(0)-r_1(0)\hat{z}_t+A^{(n-1)}+B_z^{(n-1)}A\hat{z}_t+\mu_d+C\hat{z}_t+\kappa[-\gamma(0)';\widetilde{z}]+(\kappa_1[-\gamma(0)';\widetilde{z}]\gamma_1(0)+\kappa_2[-\gamma(0);\widetilde{z}])\hat{z}_t} \times \\
&\quad \times e^{\kappa[-\gamma(0)'+V_{n-1}(0)';\widetilde{z}]+\left(\kappa_1[-\gamma(0)'+V_{n-1}(0)';z_t][-\gamma_1(0)+V_{1,n-1}(0)]+\kappa_2[-\gamma(0)'+V_{n-1}(0)';\widetilde{z}]\right)\hat{z}_t}
\end{aligned}
$$

Matching coefficients, the initial guess can be identified as the solution of matrix equation (16). It is straightforward to recombine the affine approximations of the return components to derive:

$$
\ln E_t^{\mathbb{P}}[R_{d,t+1}^{(n)}] \equiv \ln E_t^{\mathbb{P}}[e^{p_{d,t+1}^{(n-1)}-p_{d,t}^{(n)}}] = r_t + C_t(M_{t+1}, R_{d,t+1}^{(n)})
$$

Hold-to-maturity risk premia can be derived from the equilibrium expression for yields and the term structure of cashflow growth, $G_{d,t}^{(n)} \equiv E_t^{\mathbb{P}}[D_{t+n}/D_t]$, which has the recursive structure

$$
G_{d,t}^{(n)} = E_t^{\mathbb{P}}\left(\frac{D_{t+1}}{D_t} G_{d,t+1}^{(n-1)}\right)
$$

with boundary condition $G_{d,t}^{(0)} = 1$, and hence implies $G_{d,t}^{(n)} = e^{A_g^{(n)}+B_g^{(n)}\hat{z}_t}$ up to a second-order term.

D. Proof of proposition 4

Assumption (14) implies the approximate joint ccgf:

$$
\ln E_t^{\mathbb{P}}[e^{u_z'\hat{z}_{t+1}+u_d\Delta d_{t+1}}] = u'\mu + \kappa[u'\Sigma(\hat{z}_t); z_t] + u'\Phi\hat{z}_t,
$$

for $u = [u_z; u_d] \in \mathbb{R}^{n_z+1}$. We define the multiplicative martingale,

$$
Q_{t+1} = Q_t e^{-\kappa[-\gamma(\hat{z}_t);z_t]-\gamma(\hat{z}_t)'\varepsilon_{t+1}},
$$

to construct the change of measure from physical to risk-neutral probabilities, $d\mathbb{Q}/d\mathbb{P}$.

It follows that the risk-neutral dynamics of the vector process $[z; \Delta d]$ are:

$$
\begin{aligned}
\ln E_t^{\mathbb{Q}}[e^{u_z'\hat{z}_{t+1}+u_d\Delta d_{t+1}}] &= \ln E_t^{\mathbb{P}}[e^{-\kappa[-\gamma(\hat{z}_t)';z_t]-\gamma(\hat{z}_t)'\varepsilon_{t+1}+u_z'\hat{z}_{t+1}+u_d\Delta d_{t+1}}] \\
&= -\kappa[-\gamma(\hat{z}_t)'; z_t] + u_z'E_t^{\mathbb{P}}\hat{z}_{t+1} + u_d E_t^{\mathbb{P}}\Delta d_{t+1} + \mathcal{V}_t^{\mathbb{P}}(e^{-\gamma(\hat{z}_t)'\varepsilon_{t+1}+u_z'\hat{z}_{t+1}+u_d\Delta d_{t+1}}) \\
&= -\kappa[-\gamma(\hat{z}_t)'; z_t] + u'\mu + u'\Phi\hat{z}_t + \kappa[-\gamma(\hat{z}_t)' + u'\Sigma(\hat{z}_t); z_t] \\
&= \ln E_t^{\mathbb{P}}[e^{u_z'\hat{z}_{t+1}+u_d\Delta d_{t+1}}] + \kappa[-\gamma(\hat{z}_t)' + u'\Sigma(\hat{z}_t); z_t] - \kappa[-\gamma(\hat{z}_t)'; z_t] - \kappa[u'\Sigma(\hat{z}_t); z_t]
\end{aligned}
$$

E. Proof of proposition 5

The exponential-affine solution of the Perron-Frobenius eigenfunction problem can be verified:

$$\begin{aligned}\delta e^{u_z'\hat{z}_t} &= E_t^{\mathbb{P}}[e^{m_{t+1}+u_z'\hat{z}_{t+1}}] = e^{-r_t} E_t^{\mathbb{Q}}[e^{u_z'\hat{z}_{t+1}}] \\ &= e^{-r(\hat{z}_t)-\kappa[-\gamma(\hat{z}_t)';z_t]+u_z'A\hat{z}_t+\kappa[-\gamma(\hat{z}_t)'+u_z'B(\hat{z}_t);z_t]} \\ &= e^{-r(0)-r_1(0)\hat{z}_t+u_z'A\hat{z}_t+\kappa[-\gamma(0)'+V(0)';\widetilde{z}]-\kappa[-\gamma(0)';\widetilde{z}]+\kappa_1[-\gamma(0)'+V(0)';\widetilde{z}][-\gamma(0)'+V(0)']+\kappa_1[\gamma(0)';\widetilde{z}]\gamma_1(0)+\kappa_2[-\gamma(0)+V(0);\widetilde{z}]-\kappa_2[-\gamma(0);\widetilde{z}]}\end{aligned}$$

where we rely on proposition 4.

F. Additional proofs

Equivalence of Hansen-Scheinkman and Alvarez-Jermann decompositions. By proposition 3, u_z' is the equilibrium coefficient of approximate affine real bond yields, i.e., of claims to the cashflow process $d = 0$ in equation (16). Accordingly,

$$\ln(\delta) = \lim_{n\to\infty}[A_0^{(n)} - A_0^{(n-1)}] \qquad\qquad \ln[f(x_t)] = B_{0,z}^{(\infty)}\hat{z}_t$$

characterize a solution to the eigenfunction problem, where $\{A_0^{(n)}; B_{0,z}^{(n)}\}$ are the coefficients in the equilibrium expression of real bond yields. It follows that

$$m_{t+1}^T = \ln(\delta) + B_{0,z}^{(\infty)}\hat{z}_t - B_{0,z}^{(\infty)}\hat{z}_{t+1} = -r_{0,t+1}^{(\infty)}$$

which implies the equivalence.

Borovicka-Hansen elasticities. To derive the approximate expressions for shock-exposure and shock-price elasticities, define $h_{t+1}(r) \doteq r\alpha_t\varepsilon_{t+1} - \kappa[r\alpha_t; z_t]$ and note that, by the law of iterated expectations,

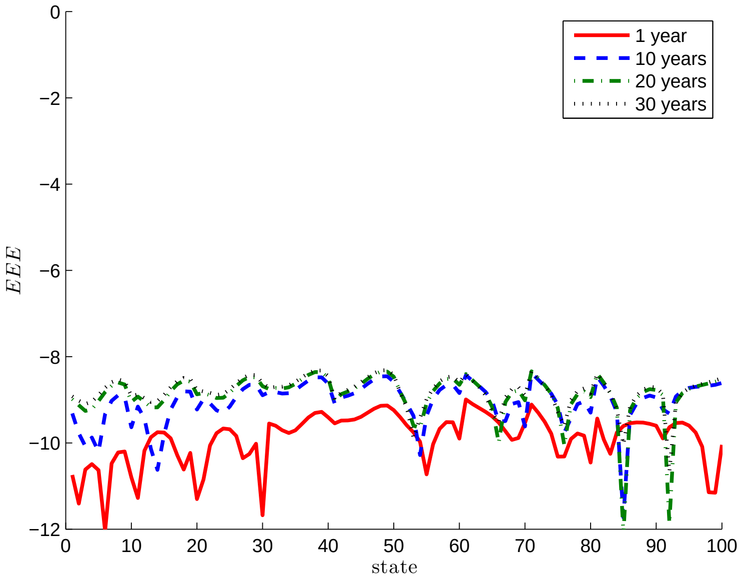
$$E_t^{\mathbb{P}}\left[e^{h_{t+1}(r)}\frac{D_{t+n}}{D_t}\right] = E_t^{\mathbb{P}}\left[e^{h_{t+1}(r)+\Delta d_{t+1}}E_{t+1}^{\mathbb{P}}\left(\frac{D_{t+n}}{D_{t+1}}\right)\right] = E_t^{\mathbb{P}}\left[e^{h_{t+1}(r)+\Delta d_{t+1}}G_{d,t+1}^{(n-1)}\right]$$

$$E_t^{\mathbb{P}}\left[e^{h_{t+1}(r)}M_{t,t+n}\frac{D_{t+n}}{D_t}\right] = E_t^{\mathbb{P}}\left[e^{h_{t+1}(r)+m_{t+1}+\Delta d_{t+1}-ny_{d,t+1}^{(n-1)}}\right]$$

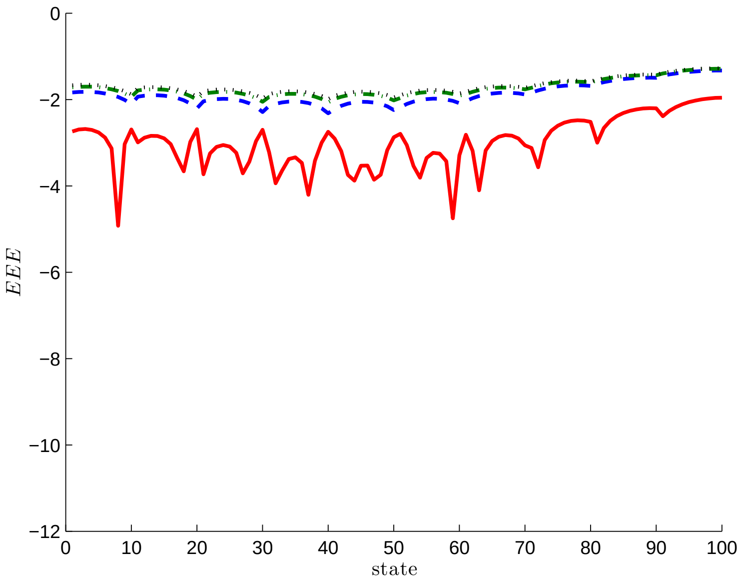
where the term structures of cashflow growth $G_{d,t}^{(n)}$ and yields $y_{d,t}^{(n)}$ have been defined above.

It follows that, under affine approximate term structures:

$$\begin{aligned}\varepsilon_{g,t} &= \frac{d}{dr}\ln E_t^{\mathbb{P}}\Big[e^{h_{t+1}(r)+\Delta d_{t+1}}F_{g,t+1}^{(n-1)}\Big]_{r=0} = \frac{d}{dr}\ln E_t^{\mathbb{P}}\Big[e^{h_{t+1}(r)+\Delta d_{t+1}+A_g^{(n-1)}+B_g^{(n-1)}\hat{z}_{t+1}}\Big]_{r=0} \\ &= \kappa_1[D(\hat{z}_t) + B_g^{(n-1)}B(\hat{z}_t); z_t]\alpha(\hat{z}_t) \\ \varepsilon_{p,t} &= \varepsilon_{g,t} - \frac{d}{dr}\ln E_t^{\mathbb{P}}\Big[e^{h_{t+1}(r)+m_{t+1}+\Delta d_{t+1}}F_{g,t+1}^{(n-1)}\Big]_{r=0} = \varepsilon_{g,t} - \frac{d}{dr}\ln E_t^{\mathbb{P}}\Big[e^{h_{t+1}(r)+m_{t+1}+\Delta d_{t+1}+A_d^{(n-1)}+B_{d,z}^{(n-1)}\hat{z}_{t+1}}\Big]_{r=0} \\ &= \kappa_1[-\gamma(\hat{z}_t) + D(\hat{z}_t) + B_z^{(n-1)}B(\hat{z}_t); z_t]\alpha(\hat{z}_t)\end{aligned}$$

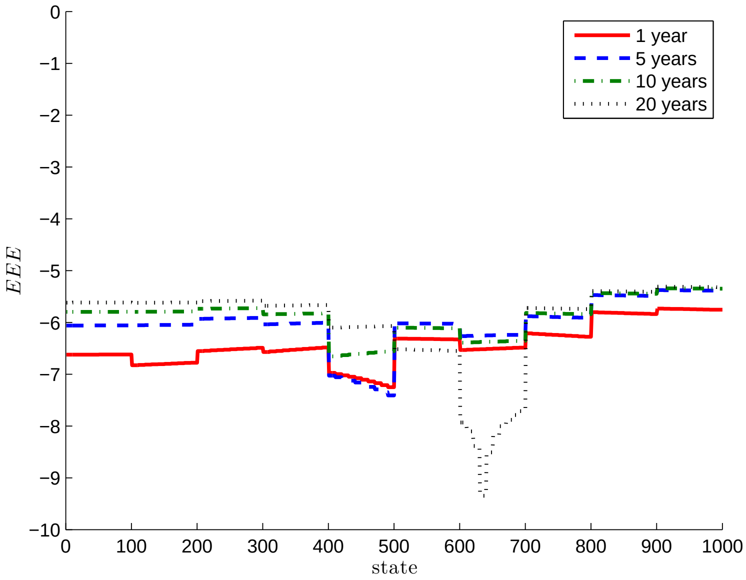


(a) *Projected solution for quantities.*

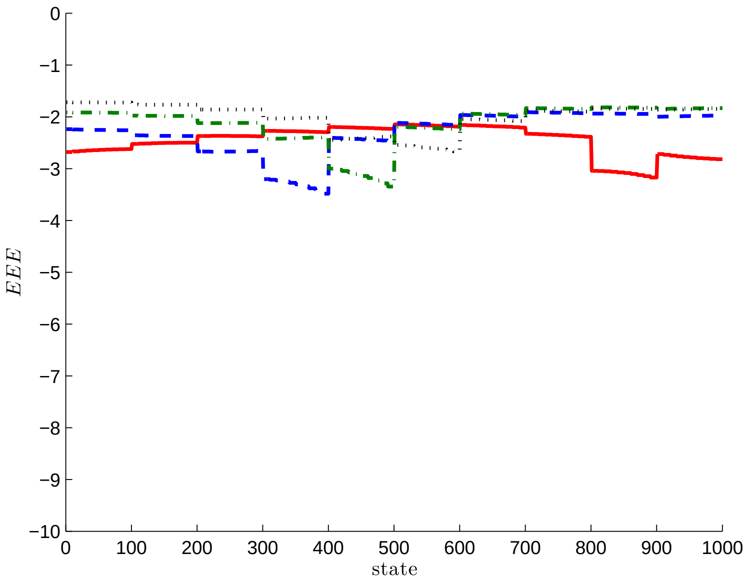


(b) *Generalized affine approximation for quantities.*

Figure 3: Multiperiod Euler equation errors in Jermann (1998) with nonlinear habits. Errors are expressed in $\log_{10}$. Values in the state dimension index different triplets $[ka_t, \sqrt{1-2\hat{s}_t}]$ built as the Cartesian product of 10 equidistant points along each dimension. The projected solution uses Chebyshev polynomials of up to degree eight collocated over a Smolyak grid. Expectations are evaluated using 10-point Gauss-Hermite quadrature.

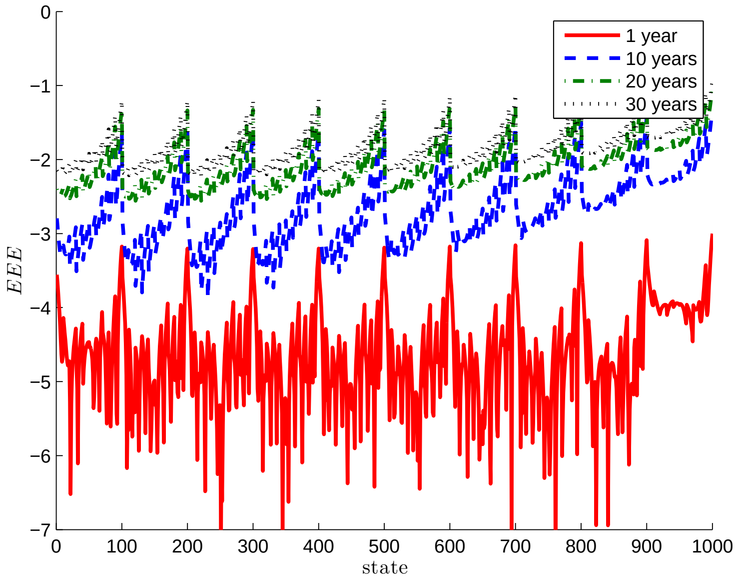


(a) *Projected solution for quantities.*

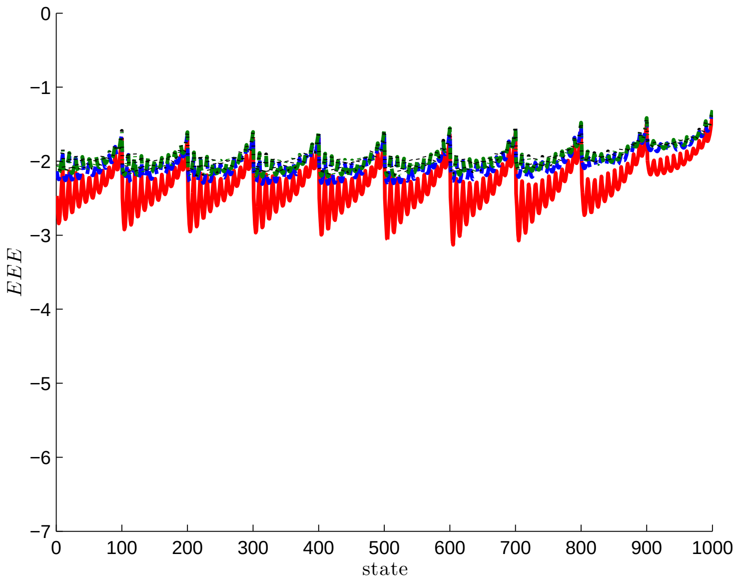


(b) *Generalized affine approximation for quantities.*

Figure 4: Multiperiod Euler equation errors in Lopez et al. (2015). Errors are expressed in $\log_{10}$. Values in the state dimension index different triplets $[u_t, \sqrt{1-2\hat{s}_t}, \Delta_{t-1}]$ built as the Cartesian product of 10 equidistant points along each dimension. The projected solution uses Chebyshev polynomials of up to degree eight collocated over a Smolyak grid. Expectations are evaluated using 10-point Gauss-Hermite quadrature.

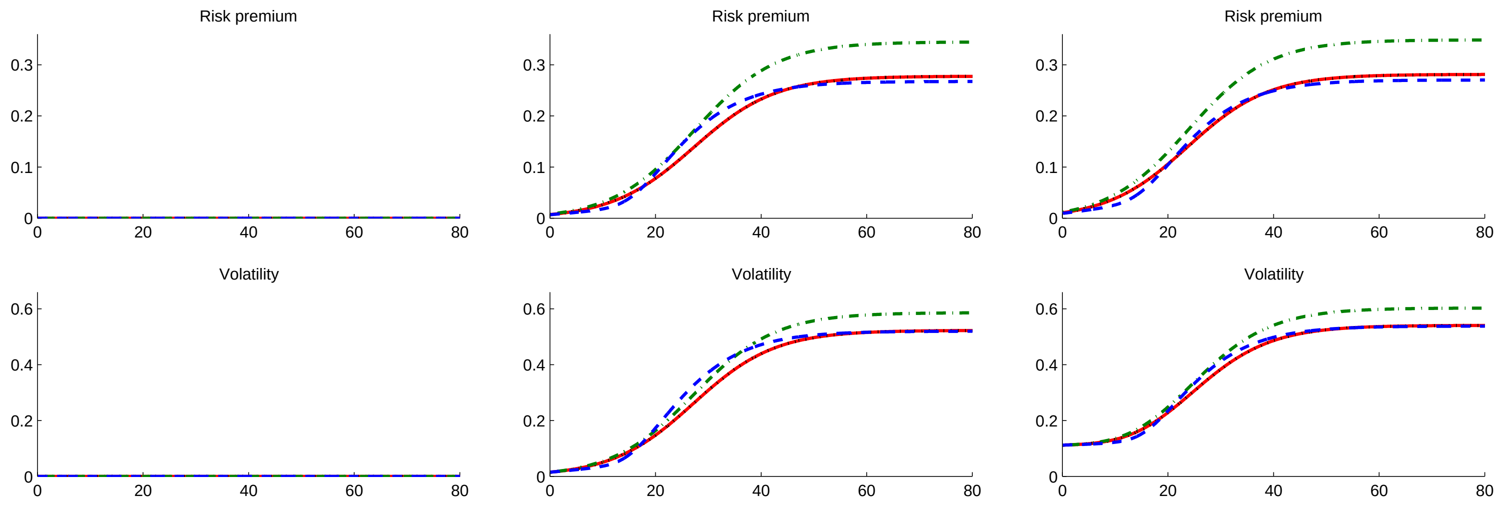


(a) *Projected solution for quantities.*



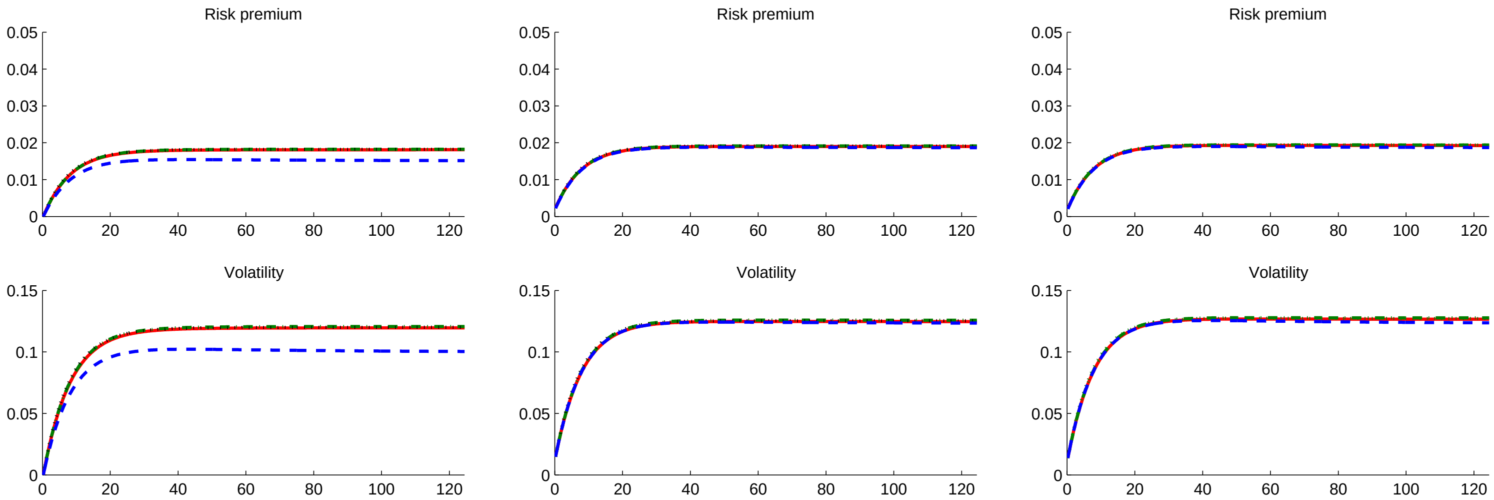
(b) *Generalized affine approximation for quantities.*

Figure 5: Multiperiod Euler equation errors in Gourio (2012). Errors are expressed in $\log_{10}$. Values in the state dimension index different triplets $[ka_t, \hat{\lambda}_t, a_t^T]$ built as the Cartesian product of 10 equidistant points along each dimension. The projected solution uses Chebyshev polynomials of up to degree eight collocated over a Smolyak grid. Expectations are evaluated using 5-point Gauss-Hermite quadrature in each 4 dimensions.



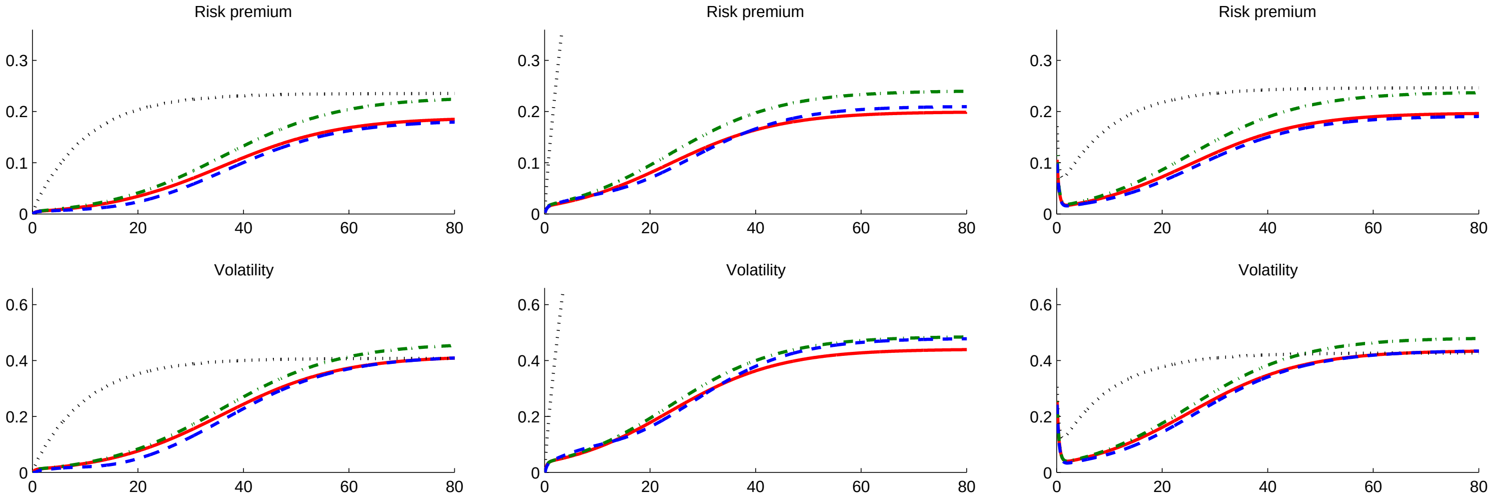
(a) *Real bonds*. (b) *Consumption strips*. (c) *Dividend strips*.

Figure 6: Comparison of solution methods to compute average equilibrium term structures of holding-period risk premia $\{\ln E_t R^{e,(n)}_{t+1}\}$ and volatilities $\{std_t(r^{(n)}_{t+1})\}$ in Campbell and Cochrane (1999). Generalized affine (solid red), perturbation around the risky steady state (dot-dashed green), standard affine (dotted black), and projected solution using cubic splines collocated over 100 Chebyshev nodes and 10-point Gauss-Hermite quadrature (dashed blue).



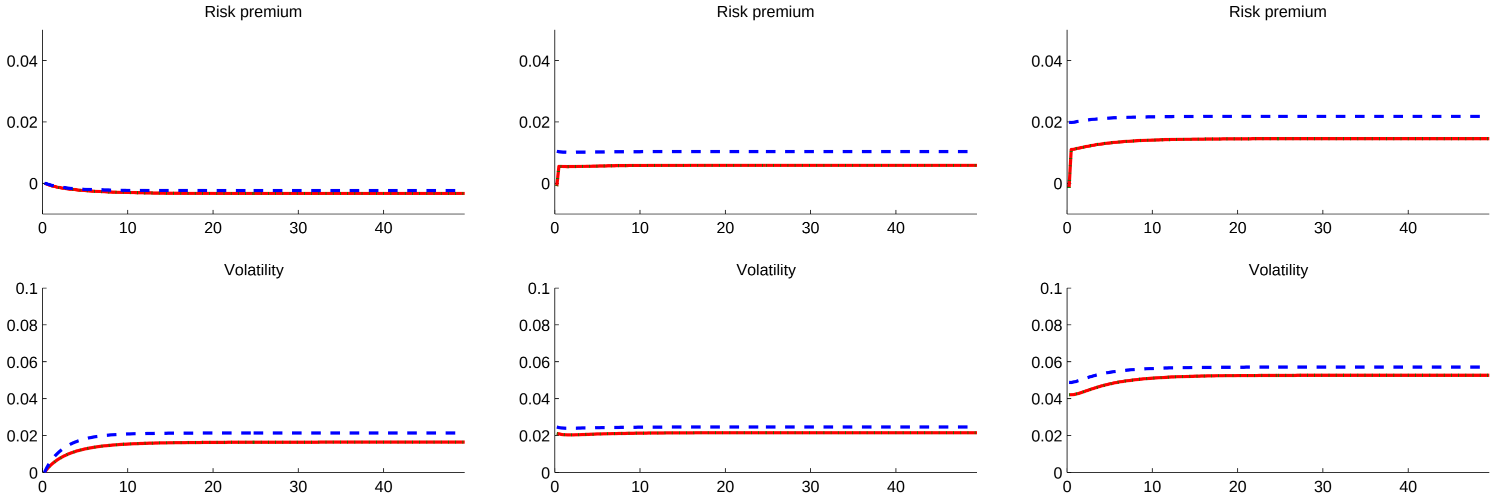
(a) *Real bonds*. (b) *Consumption strips*. (c) *Dividend strips*.

Figure 7: Comparison of solution methods to compute average equilibrium term structures of holding-period risk premia $\{\ln E_t R_{t+1}^{e,(n)}\}$ and volatilities $\{std_t(r_{t+1}^{(n)})\}$ in Jermann (1998) with nonlinear habits. Generalized affine (solid red), perturbation around the risky steady state (dot-dashed green), standard affine (dotted black), and projected solution using Chebyshev polynomials of up to degree eight collocated over a Smolyak grid and 10-point Gauss-Hermite quadrature (dashed blue).



(a) *Real bonds*. (b) *Nominal bonds*. (c) *Dividend strips*.

Figure 8: Comparison of solution methods to compute average equilibrium term structures of holding-period risk premia $\{\ln E_t R^{e,(n)}_{t+1}\}$ and volatilities $\{std_t(r^{(n)}_{t+1})\}$ in Lopez et al. (2015). Generalized affine (solid red), perturbation around the risky steady state (dot-dashed green), standard affine (dotted black), and projected solution using Chebyshev polynomials of up to degree eight collocated over a Smolyak grid and 10-point Gauss-Hermite quadrature (dashed blue).



(a) *Real bonds*. (b) *Consumption strips*. (c) *Dividend strips*.

Figure 9: Comparison of solution methods to compute average equilibrium term structures of holding-period risk premia $\{\ln E_t R^{e,(n)}_{t+1}\}$ and volatilities $\{std_t(r^{(n)}_{t+1})\}$ in Gourio (2012). Generalized affine (solid red), perturbation around the risky steady state (dashed green), standard affine (dotted black), and projected solution using Chebyshev polynomials of up to degree eight collocated over a Smolyak grid and 5-point Gauss-Hermite quadrature in each 4 dimensions (dashed blue).

ONLINE APPENDIX - NOT FOR PUBLICATION

I. Numerical considerations

The nonlinear system of equations (5) in the unknowns $[\widetilde{y}, \widetilde{z}, \widetilde{\Psi}]$ is amenable to standard numerical solution methods, yet an educated initial guess is needed to select the saddle-path stable solution. A simple continuation algorithm to solve system (5) numerically can be based on the observation that the solution $[\widetilde{y}, \widetilde{z}, \widetilde{\Psi}]$ to system:

$$0 = h(\widetilde{y}, \widetilde{z}) + f_3\widetilde{y} + f_4\widetilde{z} + q\widetilde{\mathcal{V}}(\widetilde{z}), \quad \widetilde{z} = g(\widetilde{y}, \widetilde{z})$$
$$0 = \widetilde{f}_1\widetilde{\Psi} + \widetilde{f}_2 + (f_3\widetilde{\Psi} + f_4)(\widetilde{g}_1\widetilde{\Psi} + \widetilde{g}_2) + q\widetilde{\mathcal{V}}_1(\widetilde{z})$$

coincides with a linear approximation around the deterministic steady state at $q = 0$ and around the risky steady state at $q = 1$. Accordingly, the continuation algorithm starts from the standard (i.e., saddle-path stable) linear solution at $q = 0$ and proceeds sequentially until $q = 1$ with the outcome of the previous step as the initial guess for the next. When the deterministic steady state is not a valid expansion point but the risky steady state is (as in one of the examples of section 3) the system should be solved directly at $q = 1$, so an exploration of different regions of the solution space must be carried out.

II. Comparison with third-order perturbations around the deterministic steady state

This section shows under which conditions a third-order perturbation approximation around the deterministic steady state (Schmitt-Grohé and Uribe, 2004) of the solution to dynamic system (1) *nests* the generalized affine approximation. We consider a linear specification for functions h and g: $h_{ij} = 0$, $h_{ijk} = 0$, $g_{ij} = 0$, $g_{ijk} = 0$. A nonlinear specification injects terms in second and third order perturbations that cannot be captured by the affine approximation, so this assumption of linearity is inconsequential for our purpose. For notational simplicity we assume a scalar state vector $z_t \in \mathbb{R}$; it is straightforward yet tedious to generalize the result to a multivariate state. As required in Schmitt-Grohé and Uribe (2004), we assume the time-invariance of the conditional variance of shocks: $E_t[\varepsilon_{t+1}\varepsilon'_{t+1}] = I$.

Under the implicit functions $z_{t+1} = z_{t+1}(z_t, q)$ and $y_t = y(z_t, q)$, we rewrite the dynamic system:

$$0 = E_t \exp\{h[y(z_t, q), z_t] + f_3 y[z_{t+1}(z_t, q), q] + f_4 z_{t+1}(z_t, q)\} - 1$$
$$z_{t+1}(z_t, q) = g[y(z_t, q), z_t] + \lambda(z_t)(E_{t+1} - E_t)y[z_{t+1}(z_t, q)] + q\sigma(z_t)\varepsilon_{t+1}$$

where q parametrizes the system's sensitivity to shocks. Under $q = 1$ the dynamics coincide with the original model (1). By Taylor's theorem applied at the expansion point $(z_t, q) = 0$, we have:

$$
\begin{aligned}
0 = & E_t\{([f_1y_1 + f_2 + (f_3y_1 + f_4)z_{1,t+1}]\, z_t + [f_1y_2 + f_3y_2 + (f_3y_1 + f_4)z_{2,t+1}]\, q)\} \\
& + \frac{1}{2}E_t \left\{ \begin{array}{l} ([f_1y_1 + f_2 + (f_3y_1 + f_4)z_{1,t+1}]\, z_t + [f_1y_2 + f_3y_2 + (f_3y_1 + f_4)z_{2,t+1}]\, q)^2 \\ + \left((f_3y_1 + f_4)\left[z_{11,t+1}z_t^2 + 2z_{12,t+1}z_tq + z_{22,t+1}q^2\right]\right) \\ + \left((f_1y_{11} + f_3y_{11}z_{1,t+1}^2)z_t^2 + 2(f_1y_{12} + f_3y_{12}z_{1,t+1} + f_3y_{11}z_{1,t+1}z_{2,t+1})z_tq\right) \\ +(f_1y_{22} + f_3y_{22} + 2f_3y_{12}z_{2,t+1} + f_3y_{11}z_{2,t+1}^2)q^2 \end{array} \right\} \\
& + \frac{1}{6}E_t \left\{ \begin{array}{l} ([f_1y_1 + f_2 + (f_3y_1 + f_4)z_{1,t+1}]\, z_t + [f_1y_2 + f_3y_2 + (f_3y_1 + f_4)z_{2,t+1}]\, q)^3 \\ +3\,([f_1y_1 + f_2 + (f_3y_1 + f_4)z_{1,t+1}]\, z_t + [f_1y_2 + f_3y_2 + (f_3y_1 + f_4)z_{2,t+1}]\, q) \times \\ \qquad \times \left(\begin{array}{l} (f_3y_1 + f_4)\left(z_{11,t+1}z_t^2 + 2z_{12,t+1}z_tq + z_{22,t+1}q^2\right) \\ +(f_1y_{11} + f_3y_{11}z_{1,t+1}^2)z_t^2 + 2(f_1y_{12} + f_3y_{12}z_{1,t+1} + f_3y_{11}z_{1,t+1}z_{2,t+1})z_tq \\ +(f_1y_{22} + f_3y_{22} + 2f_3y_{12}z_{2,t+1} + f_3y_{11}z_{2,t+1}^2)q^2 \end{array} \right) \\ +(f_3y_1 + f_4)\left(z_{111,t+1}z_t^3 + 3z_{112,t+1}z_t^2q + 3z_{122,t+1}z_tq^2 + z_{222,t+1}q^3\right) \\ +3f_3y_{11}(z_{1,t+1}z_t + z_{2,t+1}q)\left(z_{11,t+1}z_t^2 + 2z_{12,t+1}z_tq + z_{22,t+1}q^2\right) \\ +3f_3y_{12}\left(z_{11,t+1}z_t^2q + 2z_{12,t+1}z_tq^2 + z_{22,t+1}q^3\right) \\ +\left(f_1y_{111} + f_3y_{111}z_{1,t+1}^3\right)z_t^3 + 3\left(f_1y_{112} + f_3y_{111}z_{1,t+1}^2z_{2,t+1} + f_3y_{112}z_{1,t+1}^2\right)z_t^2q \\ +3\left(f_1y_{122} + f_3y_{122}z_{1,t+1} + f_3y_{112}z_{1,t+1}z_{2,t+1} + f_3y_{111}z_{1,t+1}z_{2,t+1}^2 \right)z_tq^2 \\ +\left(f_1y_{222} + f_3y_{222} + f_3y_{122}z_{2,t+1} + f_3y_{112}z_{1,t+1}z_{2,t+1} + f_3y_{112}z_{2,t+1}^2 + f_3y_{111}z_{2,t+1}^3 \right)q^3 \end{array} \right\}
\end{aligned}
$$

$$
\begin{aligned}
& z_{1,t+1}z_t + z_{2,t+1}q + \frac{1}{2}\left(z_{11,t+1}z_t^2 + 2z_{12,t+1}z_tq + z_{22,t+1}q^2\right) + \frac{1}{6}\left(z_{111,t+1}z_t^3 + 3z_{112,t+1}z_t^2q + 3z_{122,t+1}z_tq^2 + z_{222,t+1}q^3\right) \\
& = (g_1y_1 + g_2)z_t + g_1y_2q + (E_{t+1} - E_t)[\lambda y_1(z_{1,t+1}z_t + z_{2,t+1}q) + \lambda y_2q + \sigma\varepsilon_{t+1}q] \\
& + \frac{1}{2}g_1(y_{11}z_t^2 + 2y_{12}z_tq + y_{22}q^2) \\
& + \frac{1}{2}(E_{t+1} - E_t)\left[\begin{array}{l} 2\lambda_1y_1(z_{1,t+1}z_t^2 + z_{2,t+1}z_tq) + 2\lambda_1y_2z_tq + \lambda y_1(z_{11,t+1}z_t^2 + 2z_{12,t+1}z_tq + z_{22,t+1}q^2) \\ +2\sigma_1\varepsilon_{t+1}z_tq + \lambda y_{11}(z_{1,t+1}z_t + z_{2,t+1}q)^2 + 2\lambda y_{12}(z_{1,t+1}z_tq + z_{2,t+1}q^2) + \lambda y_{22}q^2 \end{array} \right] \\
& + \frac{1}{6}g_1(y_{111}z_t^3 + 3y_{112}z_t^2q + 3y_{122}z_tq^2 + y_{222}q^3) \\
& + \frac{1}{6}(E_{t+1} - E_t)\left[\begin{array}{l} 3\lambda_{11}y_1(z_{1,t+1}z_t^3 + z_{2,t+1}z_t^2q) + 3\lambda_{11}y_2z_t^2q \\ +\lambda y_1(z_{111,t+1}z_t^3 + 3z_{112,t+1}z_t^2q + 3z_{122,t+1}z_tq^2 + z_{222,t+1}q^3) \\ +3\lambda_1y_1(z_{11,t+1}z_t^3 + 2z_{12,t+1}z_t^2q + z_{22,t+1}z_tq^2) + 3\lambda_1y_{22}z_tq^2 \\ +3\sigma_{11}\varepsilon_{t+1}z_t^2q + \lambda y_{111}(z_{1,t+1}z_t + z_{2,t+1}q)^3 \\ +3\lambda y_{112}(z_{1,t+1}z_t + z_{2,t+1}q)^2q + 3\lambda y_{122}(z_{1,t+1}z_tq^2 + z_{2,t+1}q^3) + \lambda y_{222}q^3 \\ +3\lambda_1y_{11}z_{1,t+1}(z_{1,t+1}z_t^3 + z_{2,t+1}z_t^2q) + 3\lambda_1y_{12}(z_{1,t+1}z_t^2q + z_{2,t+1}z_tq^2) \\ +3\lambda_1y_{11}z_{2,t+1}(z_{1,t+1}z_t^2q + z_{2,t+1}z_tq^2) + 3\lambda y_{12}q(z_{11,t+1}z_t^2 + 2z_{12,t+1}z_tq + z_{22,t+1}q^2) \\ +3\lambda y_{11}(z_{1,t+1}z_t + z_{2,t+1}q)(z_{11,t+1}z_t^2 + 2z_{12,t+1}z_tq + z_{22,t+1}q^2) \end{array} \right]
\end{aligned}
$$

The approximate solution is:

$$y(z_t, q) = \underbrace{y_1 z_t + y_2 q}_{\text{1st order}} + \underbrace{\frac{1}{2}\left(y_{11} z_t^2 + 2y_{12} q z_t + y_{22} q^2\right)}_{\text{2nd order}} + \underbrace{\frac{1}{6}\left(y_{111} z_t^3 + 3y_{112} q z_t^2 + 3y_{122} q^2 z_t + y_{222} q^3\right)}_{\text{3rd order}}$$

We identify the first-order coefficients as:

$$\begin{aligned}
z_{1,t+1} &= g_1 y_1 + g_2 + \lambda y_1 (E_{t+1} - E_t) z_{1,t+1} = g_1 y_1 + g_2 \\
z_{2,t+1} &= g_1 y_2 + \lambda y_1 (E_{t+1} - E_t) z_{2,t+1} + \sigma \varepsilon_{t+1} = \sigma_z^{(1)} \varepsilon_{t+1}, \quad \sigma_z^{(1)} \doteq (I - \lambda y_1)^{-1} \sigma \\
0 &= f_1 y_1 + f_2 + (f_3 y_1 + f_4)(g_1 y_1 + g_2) \\
y_2 &= 0
\end{aligned}$$

We identify the second-order coefficients as:

$$\begin{aligned}
z_{11,t+1} &= g_1 y_{11} + 2\lambda_1 y_1 (E_{t+1} - E_t) z_{1,t+1} + \lambda y_1 (E_{t+1} - E_t) z_{11,t+1} = 0 \\
z_{12,t+1} &= g_1 y_{12} + \lambda_1 y_1 (E_{t+1} - E_t) z_{2,t+1} + \lambda y_1 (E_{t+1} - E_t) z_{12,t+1} + \sigma_1 \varepsilon_{t+1} \\
&= \sigma_z^{(2)} \varepsilon_{t+1}, \quad \sigma_z^{(2)} \doteq (I - \lambda y_1)^{-1} [\lambda_1 y_1 \sigma_z^{(1)} + \sigma_1] \\
z_{22,t+1} &= g_1 y_{22} + \lambda y_1 (E_{t+1} - E_t) z_{22,t+1} = g_1 y_{22} \\
y_{11} &= 0 \\
y_{12} &= 0 \\
0 &= [f_1 + f_3 + (f_3 y_1 + f_4) g_1] y_{22} + diag[(f_3 y_1 + f_4) \sigma_z^{(1)} \sigma_z^{(1)\prime} (f_3 y_1 + f_4)']
\end{aligned}$$

We identify the third-order coefficients as:

$$\begin{aligned}
z_{111,t+1} &= g_1 y_{111} = 0 \\
z_{112,t+1} &= g_1 y_{112} + (I - \lambda y_1)^{-1} [\lambda_{11} y_1 \sigma_z^{(1)} + 2\lambda_1 y_1 \sigma_z^{(2)} + \lambda y_{111} z_1^2 \sigma_z^{(1)} + \sigma_{11}] \varepsilon_{t+1} \\
&= (I - \lambda y_1)^{-1} [\lambda_{11} y_1 \sigma_z^{(1)} + 2\lambda_1 y_1 \sigma_z^{(2)} + \sigma_{11}] \varepsilon_{t+1} \\
z_{122,t+1} &= g_1 y_{122} \\
z_{222,t+1} &= 3(I - \lambda y_1)^{-1} \lambda y_{122} \sigma_z^{(1)} \varepsilon_{t+1} \\
y_{111} &= 0 \\
y_{112} &= 0 \\
0 &= [f_1 + (f_3 y_1 + f_4) g_1] y_{122} + f_3 y_{122} z_1 + 2diag[(f_3 y_1 + f_4) \sigma_z^{(1)} \sigma_z^{(2)\prime} (f_3 y_1 + f_4)'] \\
y_{222} &= 0
\end{aligned}$$

Therefore, the third-order solution approximated around the deterministic steady state is:

$$
\begin{aligned}
y_t &= \frac{1}{2}y_{22} + \left(y_1 + \frac{1}{2}y_{122}\right)z_t \\
z_{t+1} &= \frac{1}{2}g_1 y_{22} + \left[g_1\left(y_1 + \frac{1}{2}y_{122}\right) + g_2\right]z_t + \sigma_{z,t}\varepsilon_{t+1} \\
\sigma_{z,t} &\doteq (I - \lambda y_1)^{-1}\left\{\left[\frac{1}{2}\lambda y_{122} + \left(\lambda_1 z_t + \frac{1}{2}\lambda_{11}z_t^2\right)y_1\right]\sigma_z^{(1)} + \lambda_1 y_1 \sigma_z^{(2)} z_t^2 + \left(\sigma + \sigma_1 z_t + \frac{1}{2}\sigma_{11}z_t^2\right)\right\}
\end{aligned}
\tag{II.1}
$$

Only under special conditions the third-order perturbation (II.1) around the deterministic steady state nests the generalized affine approximation:

$$
\begin{aligned}
y_t &= \widetilde{y} + \widetilde{\Psi}(z_t - \widetilde{z}) \\
z_{t+1} &= g(\widetilde{y}, \widetilde{z}) + (\widetilde{g}_1\widetilde{\Psi} + \widetilde{g}_2)(z_t - \widetilde{z}) + [I_{n_z} - \lambda(z_t)\widetilde{\Psi}]^{-1}\sigma(z_t)\varepsilon_{t+1} \\
0 &= h(\widetilde{y}, \widetilde{z}) + f_3\widetilde{y} + f_4\widetilde{z} + \kappa\left[(f_3\widetilde{\Psi} + f_4)[I_{n_z} - \lambda(z)\widetilde{\Psi}]^{-1}\sigma(z); z\right] \\
0 &= \widetilde{f_1}\widetilde{\Psi} + \widetilde{f_2} + (f_3\widetilde{\Psi} + f_4)(\widetilde{g}_1\widetilde{\Psi} + \widetilde{g}_2) + \frac{\partial}{\partial z}\kappa\left[(f_3\widetilde{\Psi} + f_4)[I_{n_z} - \lambda(z)\widetilde{\Psi}]^{-1}\sigma(\widetilde{z}); \widetilde{z}\right]
\end{aligned}
$$

In particular, the nesting property holds for the modeling of y_t and $E_t z_{t+1}$ if: i) shocks are Gaussian; ii) function g is linear in both its arguments; iii) function h is linear in both its arguments; iv) $\kappa[(f_3\widetilde{\Psi} + f_4)[I_{n_z} - \lambda(z)\widetilde{\Psi}]^{-1}\sigma(z); z] = \kappa[(f_3 y_1 + f_4)[I_{n_z} - \lambda(z)y_1]^{-1}\sigma(z); z]$.

Finally, the modeling of innovations under the generalized affine approximation, $(E_{t+1} - E_t)z_{t+1}$ is nested in third-order perturbations around the deterministic steady state if λ and σ are state-independent.

III. Comparison with Coeurdacier et al. (2011)

Following Coeurdacier et al. (2011), the extant practice for linear perturbations around the risky steady state relies on a second-order approximation of the risky steady state. This procedure starts by approximating the original problem around $y_{t+1} = E_t y_{t+1}$ and $z_{t+1} = E_t z_{t+1}$ as:

$$
\begin{aligned}
0 &= h(y_t, z_t) + f_3 E_t y_{t+1} + f_4 E_t z_{t+1} + \mathcal{V}_t\left(e^{f_3 y_{t+1} + f_4 z_{t+1}}\right) \\
&\approx h(y_t, z_t) + f_3 E_t y_{t+1} + f_4 E_t z_{t+1} + E_t\left[f_3(y_{t+1} - E_t y_{t+1}) + f_4(z_{t+1} - E_t z_{t+1})\right] \\
&\quad + \frac{1}{2}E_t\left[(f_3(y_{t+1} - E_t y_{t+1}) + f_4(z_{t+1} - E_t z_{t+1}))^2\right] - \frac{1}{2}\left(E_t\left[f_3(y_{t+1} - E_t y_{t+1}) + f_4(z_{t+1} - E_t z_{t+1})\right]\right)^2 \\
&= h(y_t, z_t) + f_3 E_t y_{t+1} + f_4 E_t z_{t+1} + \frac{1}{2}var_t\left(f_3 y_{t+1} + f_4 z_{t+1}\right)
\end{aligned}
$$

One then conjectures a linear solution $y_t = \overline{y} + \overline{\Psi}(z_t - \overline{z})$ and verifies it by solving the resulting system. This procedure boils down in our context to replacing entropy with variance. This strategy works under normal shocks but does not under any other distributions whose entropy does not coincide with half the variance. In fact, we showed that approximating this problem is unnecessary.