Entropy-based approximations of DSGE models: A unified theory of risk-adjusted linearizations

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MMCI Research Conference 2017

Intro

Variation in risk sensitivity is key in many dynamic equilibrium models (time-varying risk aversion, stochastic volatility, disaster risk, ...)

How to solve for equilibrium quantities and asset prices?

Conventional linear perturbations are easy to compute, provide intuition, and have precise mathematical foundation but do not capture risk premia $\Rightarrow$ asset pricing and welfare costs of fluctuations cannot be studied.

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Global solutions are accurate but computationally intensive and offer limited analytic insight.

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Conventional higher-order perturbations get analytically intractable fast; non-analytic functions; determinisic SS may be invalid expansion point.

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Finance – At least since the 1990s (Hansen, Campbell, Jermann, ...): risk corrections based on ad-hoc loglinear-lognormal/affine approximations.

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Macro – Last decade: linear perturbations around risky steady state (Coeurdacier-Rey-Winant, Juillard, Meyer-Gohde).

This paper

Affine approximations used heuristically in finance can be generalized to:

- Handle non-Gaussian shocks
- Acquire mathematical foundation

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- Handle non-Gaussian shocks
- Acquire mathematical foundation $\Rightarrow$ equivalence with linear perturbation around the risky steady state (SS).

Risky SS: ex-post shocks are 0 but agents don't know that ex ante (Juillard, 2011; Coeurdacier et al., 2011).

Deterministic SS: ex-post shocks are 0 and agents know that ex ante.

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Affine approximations used heuristically in finance can be generalized to:

- Handle non-Gaussian shocks
- Acquire mathematical foundation $\Rightarrow$ equivalence with linear perturbation around the risky steady state (SS).
  $\Rightarrow$ closed-form formulas for equilibrium coefficients
  $\Rightarrow$ counterpart to Blanchard-Kahn conditions

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Deterministic SS: ex-post shocks are 0 and agents know that ex ante.

Example 1: Campbell-Cochrane in an endowment economy

Suppose the economy is described by:

$$
\begin{aligned}
0 &= \ln E_t e^{m_{t+1}+r_t} \\
m_{t+1} &= \ln(\beta) - \gamma \Delta c_{t+1} - \gamma \Delta s_{t+1} \\
s_{t+1} &= \phi s_t + \lambda(s_t)\sigma\varepsilon_{t+1} \\
c_{t+1} &= c_t + \sigma\varepsilon_{t+1}
\end{aligned}
$$

with $\varepsilon_t \sim Niid(0,1)$.

Example 1: Campbell-Cochrane in an endowment economy

Suppose the economy is described by:

$$
\begin{aligned}
0 &= \ln e^{E_t m_{t+1} + \frac{1}{2} var_t(m_{t+1}) + r_t} \\
m_{t+1} &= \ln(\beta) - \gamma \Delta c_{t+1} - \gamma \Delta s_{t+1} \\
s_{t+1} &= \phi s_t + \lambda(s_t) \sigma \varepsilon_{t+1} \\
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Example 1: Affine approximation

We conjecture $r_t = r + \psi_{rs} s_t$:

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& = \ln\beta + \gamma(1-\phi)s_t + r + \psi_{rs} s_t + \frac{\gamma^2 [1+\lambda(s_t)]^2 \sigma^2}{2} \\
& \approx \ln\beta + \frac{\gamma^2 [1+\lambda(0)]^2 \sigma^2}{2} + r + \gamma(1-\phi)s_t + \psi_{rs} s_t + \frac{\gamma^2 [1+\lambda(0)]\lambda'(0)\sigma^2}{2} s_t
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and solve for unknown coefficients r, ψ_{rc} and ψ_{rs}:

$$r = -\ln\beta - \underbrace{\frac{\gamma^2[1+\lambda(0)]^2\sigma^2}{2}}_{\text{precautionary saving}}, \quad \psi_{rs} = -\underbrace{\gamma(1-\phi)}_{\substack{\text{intertemporal} \\ \text{substitution}}} - \underbrace{\gamma[1+\lambda(0)]\lambda'(0)\sigma^2}_{\text{precautionary saving}}$$

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Let's generalize this line of reasoning...

Let the dynamic system:

$$0 = \ln E_t \exp[h(y_t, z_t) + f_3 y_{t+1} + f_4 z_{t+1}]$$
$$z_{t+1} = g(y_t, z_t) + \lambda(z_t)(E_{t+1} - E_t)y_{t+1} + \sigma(z_t)\varepsilon_{t+1}$$

wher $\varepsilon_t \sim MDS$, with conditional cumulant generating function

$$\kappa[\alpha(z_t); z_t] \doteq \ln E_t \exp[\alpha(z_t)'\varepsilon_{t+1}]$$

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z_{t+1} &= g(y_t, z_t) + \lambda(z_t)(E_{t+1} - E_t) y_{t+1} + \sigma(z_t)\varepsilon_{t+1}
\end{aligned}
$$

wher $\varepsilon_t \sim MDS$, with conditional cumulant generating function

$$\kappa[\alpha(z_t); z_t] \doteq \ln E_t \exp[\alpha(z_t)' \varepsilon_{t+1}]$$

We define the conditional, relative entropy of a random variable x by

$$\mathcal{V}_t[\exp(x_{t+1})] \doteq \ln E_t \exp(x_{t+1}) - E_t x_{t+1}$$

It's a non-negative measure of dispersion. If $x \sim N(0, \sigma^2)$, then $\mathcal{V}(e^x) = \frac{\sigma^2}{2}$

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$$z_{t+1} = g(y_t, z_t) + \lambda(z_t)(E_{t+1} - E_t) y_{t+1} + \sigma(z_t)\varepsilon_{t+1}$$

wher $\varepsilon_t \sim MDS$, with conditional cumulant generating function

$$\kappa[\alpha(z_t); z_t] \doteq \ln E_t \exp[\alpha(z_t)' \varepsilon_{t+1}]$$

We are looking for an affine solution with unknowns $[\widetilde{y}; \widetilde{z}; \widetilde{\Psi}]$:

$$y_t = \widetilde{y} + \widetilde{\Psi}(z_t - \widetilde{z})$$

hence: $z_{t+1} = g(y_t, z_t) + \widetilde{\sigma}_z(z_t)\varepsilon_{t+1}, \quad \widetilde{\sigma}_z(z_t) \doteq [I_{n_z} - \lambda(z_t)\widetilde{\Psi}]^{-1}\sigma(z_t)$

$$\widetilde{\mathcal{V}}(z_t) \doteq \mathcal{V}_t[\exp((f_3 \widetilde{\Psi} + f_4) z_{t+1})] = \kappa[(f_3 \widetilde{\Psi} + f_4)\widetilde{\sigma}_z(z_t); z_t]$$

Let's generalize this line of reasoning...

Given system:

$$
\begin{aligned}
0 &= \ln E_t \exp[h(y_t, z_t) + f_3 y_{t+1} + f_4 z_{t+1}] \\
z_{t+1} &= g(y_t, z_t) + \lambda(z_t)(E_{t+1} - E_t)y_{t+1} + \sigma(z_t)\varepsilon_{t+1}
\end{aligned}
\tag{1}
$$

the affine approximate solution is:

$$
\begin{aligned}
y_t &= \widetilde{y} + \widetilde{\Psi}(z_t - \widetilde{z}) \\
z_{t+1} &= \widetilde{z} + \widetilde{g}_1(y_t - \widetilde{y}) + \widetilde{g}_2(z_t - \widetilde{z}) + [I_{n_z} - \lambda(z_t)\widetilde{\Psi}]^{-1}\sigma(z_t)\varepsilon_{t+1}
\end{aligned}
$$

where the unknowns $[\widetilde{y}, \widetilde{z}, \widetilde{\Psi}]$ solve the system of equations:

$$
\begin{aligned}
\widetilde{z} &= g(\widetilde{y}, \widetilde{z}) \\
0 &= h(\widetilde{y}, \widetilde{z}) + f_3\widetilde{y} + f_4\widetilde{z} + \widetilde{\mathcal{V}}(\widetilde{z}) \\
0 &= \widetilde{f}_1\widetilde{\Psi} + \widetilde{f}_2 + (f_3\widetilde{\Psi} + f_4)(\widetilde{g}_1\widetilde{\Psi} + \widetilde{g}_2) + \widetilde{\mathcal{V}}_1(\widetilde{z})
\end{aligned}
\tag{2}
$$

with $\widetilde{\mathcal{V}}(z_t) \doteq \kappa\left[(f_3\widetilde{\Psi} + f_4)[I_{n_z} - \lambda(z_t)\widetilde{\Psi}]^{-1}\sigma(z_t); z_t\right]$.

Affine approximation: Formal derivation

Existence and uniqueness of saddle-path solution grant application of the implicit function theorem. Adapt Blanchard-Kahn (1980) conditions.

Proposition

The point $(y_t, z_t, q) = (\tilde{y}, \tilde{z}, 1)$ is a saddle point if and only if the generalized eigenvalues

$$\alpha(\Gamma, \Xi) \doteq \{\alpha \in \mathbb{C} : \det(\Gamma\alpha - \Xi) = 0\} = \{\alpha_i, i = 1, ..., n_y + n_z\}$$

of the square matrices:

$$\Gamma \doteq \begin{bmatrix} f_4 & f_3 \\ I_{n_z} & 0 \end{bmatrix} \qquad \Xi \doteq \begin{bmatrix} -f_2(\tilde{y}, \tilde{z}) - \tilde{\mathcal{V}}_1(\tilde{z}) & -f_1(\tilde{y}, \tilde{z}) \\ g_2(\tilde{y}, \tilde{z}) & g_1(\tilde{y}, \tilde{z}) \end{bmatrix}$$

are such that there are n_z generalized eigenvalues with modulus within the unit circle and n_y with modulus larger than unity.

Affine approximation: Formal derivation

Consider the parametrized family of system (1):

$$0 = E_t x^q_{t+1} + \nu^q_t, \quad \nu^q_t \doteq \mathcal{V}_t[\exp(x^q_{t+1})]$$
$$z^q_{t+1} = g[y(z_t, q), z_t] + \lambda(z_t)(E_{t+1} - E_t)y[z(z_t, q, q\varepsilon_{t+1}), q] + q\sigma(z_t)\varepsilon_{t+1}$$
$$x^q_{t+1} = h[y(z_t, q), z_t] + f_3 y[z(z_t, q, q\varepsilon_{t+1}), q] + f_4 z(z_t, q, q\varepsilon_{t+1})$$

where the perturbation scalar $q \in [0, 1]$ indexes the system's sensitivity to shocks, so that under $q = 1$ the dynamics coincide with (1).
Let $\bar{z} = z(\bar{z}, 1, 0)$ a risky steady state.

Proposition

Suppose that a risky steady state $(z_t, q) = (\bar{z}, 1)$ is a saddle point. Then, the affine approximate solution locally unique and stable, and the solution $[\widetilde{y}, \widetilde{z}, \widetilde{\Psi}]$ to system (2) coincides with the coefficients from a linear perturbation around the risky steady state:

$$\widetilde{y} = y(\bar{z}, 1), \qquad \widetilde{z} = \bar{z}, \qquad \widetilde{\Psi} = y_1(\bar{z}, 1)$$

Affine approximation and conventional perturbations

Affine approximations are not nested in conventional perturbations around the deterministic steady state.
We can at most reconstruct implicit functions $y(0,q)$ and $y_1(0,q)$ using output from N-order perturbations around $(z_t, q) = (0,0)$ as:

$$y(0,q) = \lim_{N\to\infty} \sum_{i=1}^{N} \frac{1}{i!} \frac{\partial^i y(0,q)}{\partial q^i}\Big|_{q=0} q^i, \quad y_1(0,q) = \lim_{N\to\infty} \sum_{i=0}^{N} \frac{1}{i!} \frac{\partial^i y_1(0,q)}{\partial q^i}\Big|_{q=0} q^i$$

as long as $y(0,q)$ and $y_1(0,q)$ have convergent Taylor series at $q = 0$ with a sufficiently large radius of convergence. Functions of interest: $y(\bar{z},1)$ and $y_1(\bar{z},1)$.

Proposition

If $y(\bar{z},1) \neq y(0,1)$ or $y_1(\bar{z},1) \neq y_1(0,1)$, then risky steady state perturbations are not nested in deterministic steady state perturbations of arbitrary order N.

Key examples

- Campbell–Cochrane (1999)
- Wachter (2013)
- Coeurdacier–Rey–Winant (2011)
- Jermann (1998)
- Lopez–Lopez-Salido–Vazquez-Grande (2015)
- Gourio (2012)

Campbell-Cochrane (1999) endowment economy

Campbell-Cochrane (1999): Pricing wealth

Log price-consumption ratio of wealth portfolio solves the forward-looking difference equation:

$$
\begin{aligned}
e^{pc_t} &= E_t e^{\ln(\beta)+(1-\gamma)\Delta c_{t+1}-\gamma\Delta s_{t+1}} + E_t e^{\ln(\beta)+(1-\gamma)\Delta c_{t+1}-\gamma\Delta s_{t+1}+pc_{t+1}} \\
&= E_t e^{\ln(\beta)+(1-\gamma)\Delta c_{t+1}-\gamma\Delta s_{t+1}+\ln(1+e^{pc_{t+1}})}
\end{aligned}
$$

When combined with a terminal condition that rules out bubbles, the forward-looking difference equation unwinds as:

$$
pc_t = \ln\left(\sum_{n=1}^{\infty} e^{pc_t^{(n)}}\right), \quad pc_t^{(n)} = \ln E_t e^{\ln(\beta)+(1-\gamma)\Delta c_{t+1}-\gamma\Delta s_{t+1}+pc_{t+1}^{(n-1)}}
$$

where $pc_t^{(n)}$ describes the log price-consumption ratio of the nth consumption strip, with $pc_t^{(0)} = 0$.

Campbell-Cochrane (1999): Pricing wealth

Step 1. Split into certainty-equivalent and entropy terms:

$$0 = \ln(\beta) + (1-\gamma)E_t\Delta c_{t+1} - \gamma E_t \Delta s_{t+1} + E_t pc_{t+1}^{(n-1)} - pc_t^{(n)} + \mathcal{V}_t\left(e^{(1-\gamma)\Delta c_{t+1} - \gamma\Delta s_{t+1} + pc_{t+1}^{(n-1)}}\right)$$

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Step 2. Guess $pc_t^{(n)} = \widetilde{pc}^{(n)} + \widetilde{\psi}^{(n)}\widehat{s}_t$, hence:

$$\mathcal{V}_t\left(e^{(1-\gamma)\Delta c_{t+1} - \gamma\Delta s_{t+1} + pc_{t+1}^{(n-1)}}\right) = \left(1 - \gamma[1 + \Lambda(\widehat{s}_t)] + \widetilde{\psi}^{(n-1)}\Lambda(\widehat{s}_t)\right)^2 \frac{\sigma^2}{2}$$

Campbell-Cochrane (1999): Pricing wealth

Step 3.

$$0 = \ln(\beta) + (1-\gamma)E_t\Delta c_{t+1} - \gamma E_t\Delta s_{t+1} + E_t pc_{t+1}^{(n-1)} - pc_t^{(n)}$$
$$+ \left(1 - \gamma[1+\Lambda(\widehat{s}_t)] + \widetilde{\psi}^{(n)}\Lambda(\widehat{s}_t)\right)^2 \frac{\sigma^2}{2}$$

Campbell-Cochrane (1999): Pricing wealth

Step 3. Linearize:

$$
\begin{aligned}
0 &= \ln(\beta) + (1-\gamma)E_t\Delta c_{t+1} - \gamma E_t \Delta s_{t+1} + E_t pc_{t+1}^{(n-1)} - pc_t^{(n)} \\
&\quad + \left(1 - \gamma[1 + \Lambda(\widehat{s}_t)] + \widetilde{\psi}^{(n)}\Lambda(\widehat{s}_t)\right)^2 \frac{\sigma^2}{2} \\
&\approx \widetilde{pc}^{(n-1)} - \widetilde{pc}^{(n)} + \ln(\beta e^{(1-\gamma)\mu}) + \left(1 - \gamma[1+\Lambda(0)] + \widetilde{\psi}^{(n-1)}\Lambda(0)\right)^2 \frac{\sigma^2}{2} \\
&\quad + \widetilde{\psi}^{(n-1)}\phi\widehat{s}_t - \widetilde{\psi}^{(n)}\widehat{s}_t + \gamma(1-\phi)\widehat{s}_t \\
&\quad + \left(1 - \gamma[1+\Lambda(0)] + \widetilde{\psi}^{(n-1)}\Lambda(0)\right)(\widetilde{\psi}^{(n-1)} - \gamma)\Lambda_1(0)\sigma^2\widehat{s}_t
\end{aligned}
$$

Campbell-Cochrane (1999): Pricing wealth

Step 4. Identify:

$$\widetilde{pc}^{(n)} = \widetilde{pc}^{(n-1)} + \ln(\beta e^{(1-\gamma)\mu}) + \left(1 - \gamma[1 + \Lambda(0)] + \widetilde{\psi}^{(n-1)}\Lambda(0)\right)^2 \frac{\sigma^2}{2}$$

$$\widetilde{\psi}^{(n)} = \widetilde{\psi}^{(n-1)}\phi + \gamma(1-\phi) + \left(1 - \gamma[1 + \Lambda(0)] + \widetilde{\psi}^{(n-1)}\Lambda(0)\right)(\widetilde{\psi}^{(n-1)} - \gamma)\Lambda_1(0)\sigma^2$$

with boundary condition $\widetilde{pc}^{(0)} = \widetilde{\psi}^{(0)} = 0$. The approximate solution is:

$$pc_t = \ln\left(\sum_{n=1}^{\infty} e^{pc_t^{(n)}}\right), \quad pc_t^{(n)} = \widetilde{pc}^{(n)} + \widetilde{\psi}^{(n)}\widehat{s}_t$$

Campbell-Cochrane (1999): Pricing wealth

Compare to a conventional third-order perturbation:

$$\widehat{s}_{t+1} = \phi \widehat{s}_t + \Lambda(0)\sigma\varepsilon_{t+1} + \underbrace{\Lambda_1(0)\widehat{s}_t\sigma\varepsilon_{t+1}}_{\text{2nd order}} + \underbrace{\frac{1}{2}\Lambda_{11}(0)\widehat{s}_t^2\sigma\varepsilon_{t+1}}_{\text{3rd order}}$$

$$pc_t = \ln\left(\sum_{n=1}^{\infty} e^{pc_t^{(n)}}\right), \quad pc_t^{(n)} = n\ln(\beta e^{(1-\gamma)\mu}) + \bar{\psi}_1^{(n)}\widehat{s}_t + \bar{\psi}_2^{(n)} + \bar{\psi}_3^{(n)}\widehat{s}_t$$

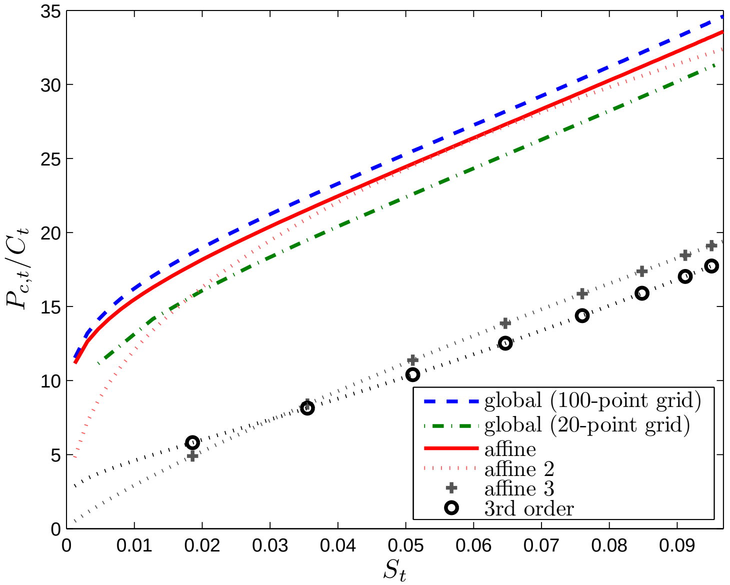
where:

$$\bar{\psi}_1^{(n)} = \gamma(1-\phi^n)$$

$$\bar{\psi}_2^{(n)} = \bar{\psi}_2^{(n-1)} + \left(1-\gamma[1+\Lambda(0)] + \bar{\psi}_1^{(n-1)}\Lambda(0)\right)^2\frac{\sigma^2}{2}$$

$$\bar{\psi}_3^{(n)} = \bar{\psi}_3^{(n-1)}\phi + \left(1-\gamma[1+\Lambda(0)] + \bar{\psi}_1^{(n-1)}\Lambda(0)\right)(\bar{\psi}_1^{(n-1)} - \gamma)\Lambda_1(0)\sigma^2$$

with boundary conditions $\bar{\psi}_2^{(0)} = \bar{\psi}_3^{(0)} = 0$.


35
30
25
20
15
10
5
0
$P_{c,t}/C_t$
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
S_t
global (100-point grid)
global (20-point grid)
affine
affine 2
affine 3
3rd order

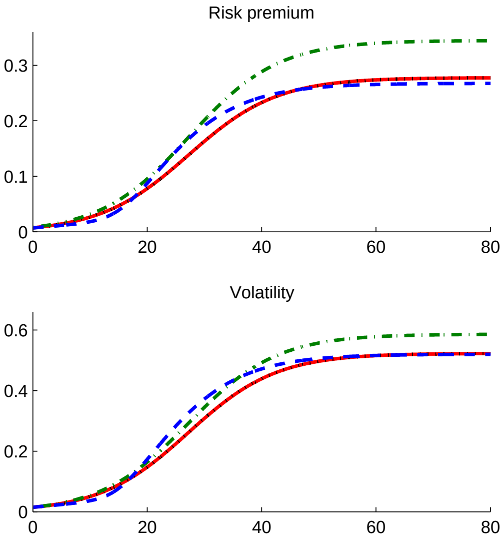


Figure: Consumption strips in Campbell-Cochrane (1999)

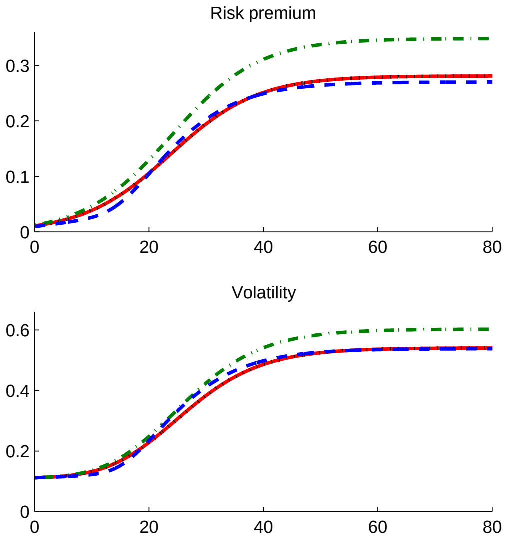


Figure: Dividend strips in Campbell-Cochrane (1999)

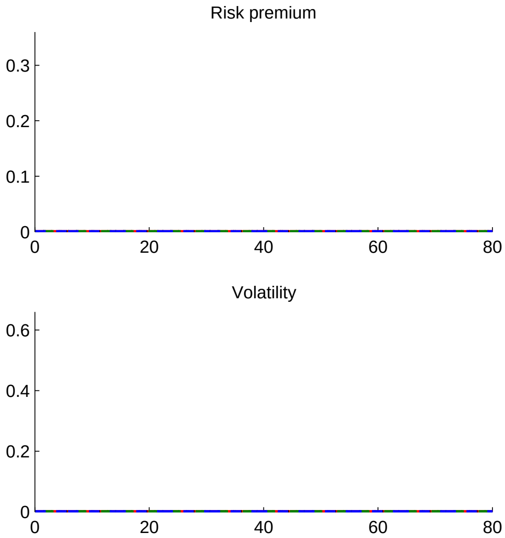
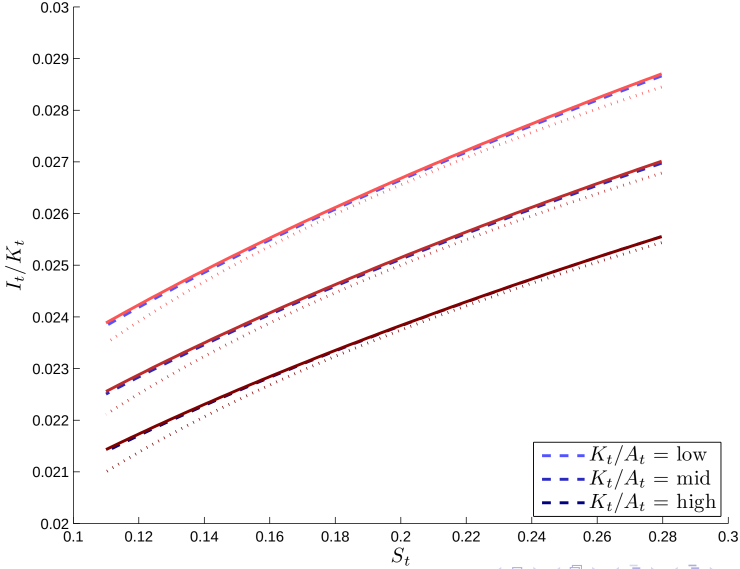


Figure: Real bonds in Campbell-Cochrane (1999)

Wachter (2013) endowment economy with recursive preferences and variable disaster risk.

Coeurdacier-Rey-Winant (2011) small open economy with intertemporal consumption choice.

Jermann (1998) production economy with capital adjustment costs and Campbell-Cochrane habits


0.03
0.029
0.028
0.027
0.026
0.025
0.024
0.023
0.022
0.021
0.02
I_t/K_t
0.1
0.12
0.14
0.16
0.18
0.2
0.22
0.24
0.26
0.28
0.3
S_t
K_t/A_t = low
K_t/A_t = mid
K_t/A_t = high

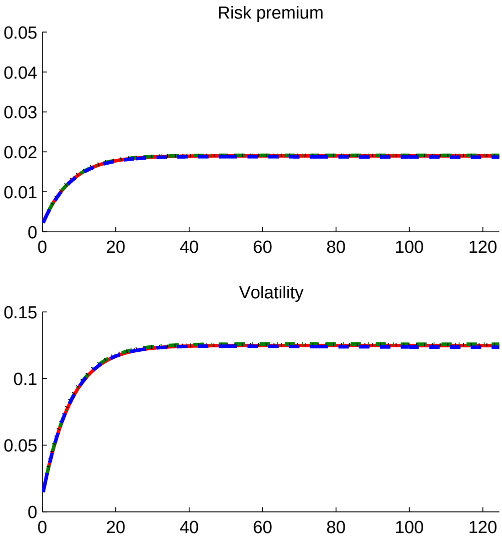


Figure: Consumption strips in Jermann (1998) with nonlinear habits

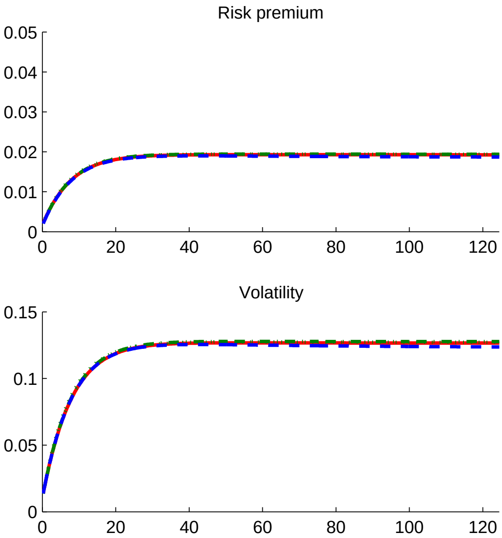


Figure: Dividend strips in Jermann (1998) with nonlinear habits

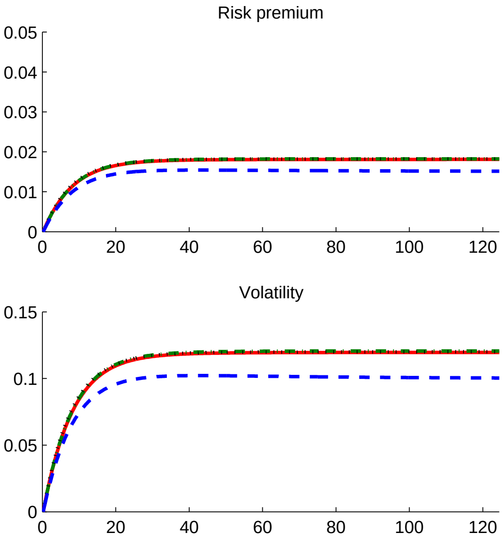


Figure: Real bonds in Jermann (1998) with nonlinear habits

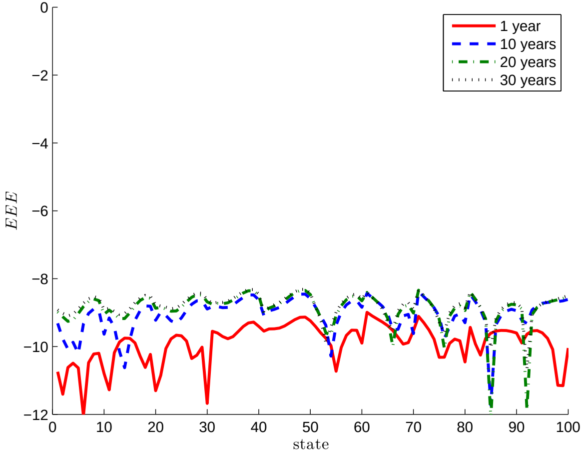


Figure: EEE in Jermann (1998) with CC habits. Projected solution

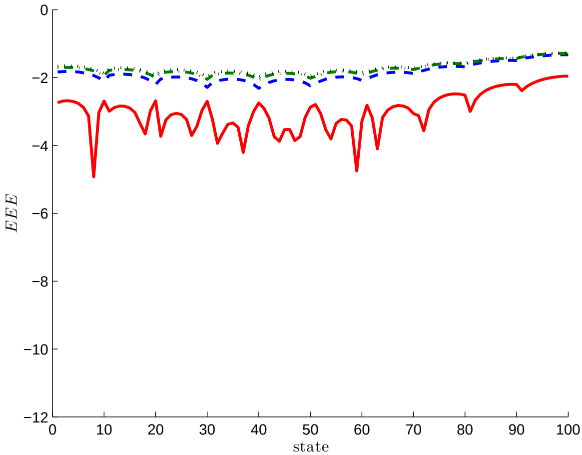


Figure: EEE in Jermann (1998) with nonlinear habits. Affine approximation

Lopez–Lopez-Salido–Vazquez-Grande (2015) New Keynesian economy with Campbell-Cochrane habits

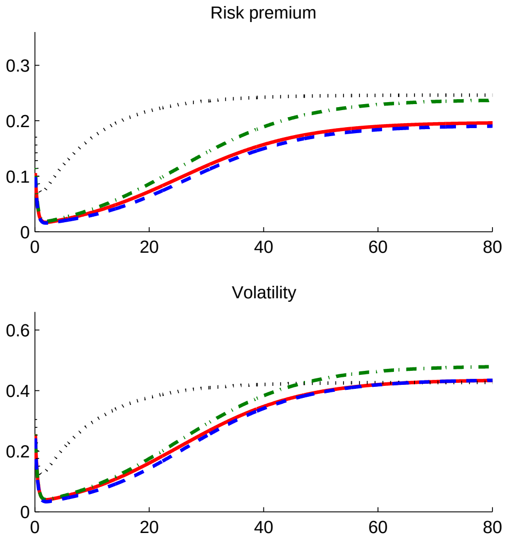


Figure: Dividend strips in Lopez et al. (2015)

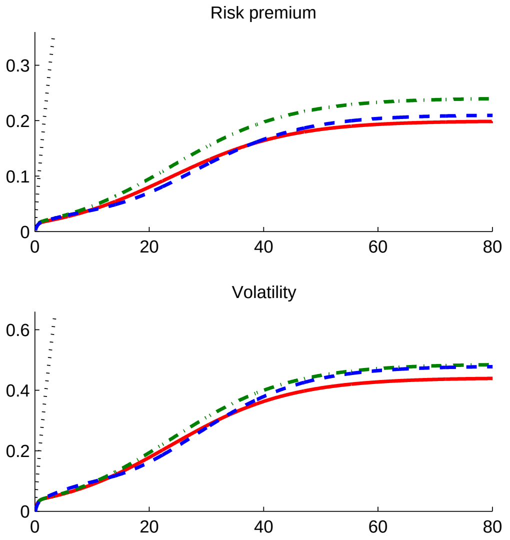


Figure: Nominal bonds in Lopez et al. (2015)

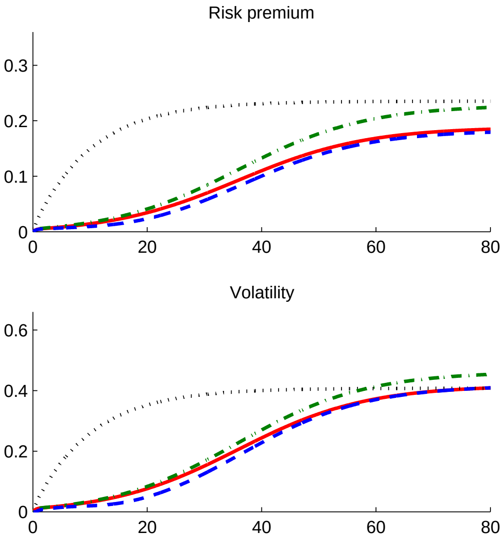


Figure: Real bonds in Lopez et al. (2015)

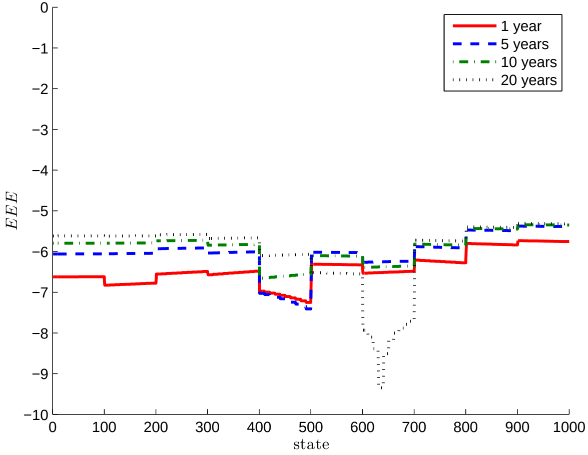


Figure: Euler Equation Errors in Lopez et al. (2015). Projected solution

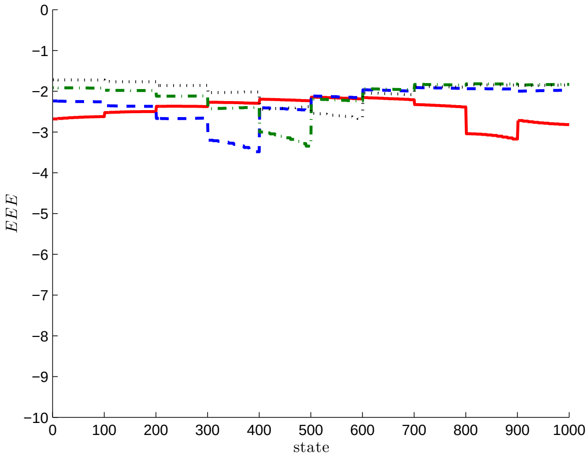


Figure: Euler Equation Errors in Lopez et al. (2015). Affine approximation

Gourio (2012) production economy with recursive preferences and variable disaster risk

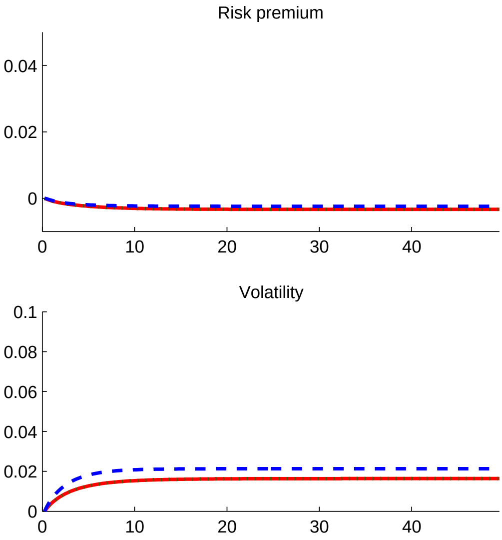


Figure: Real bonds ($d_t = 0$) in Gourio (2012)

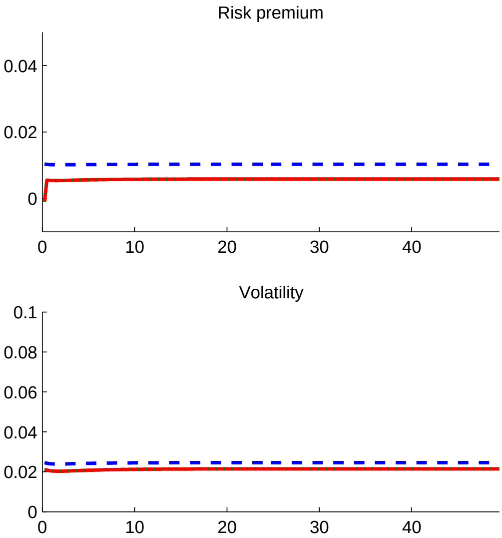


Figure: Consumption strips ($d_t = c_t$) in Gourio (2012)

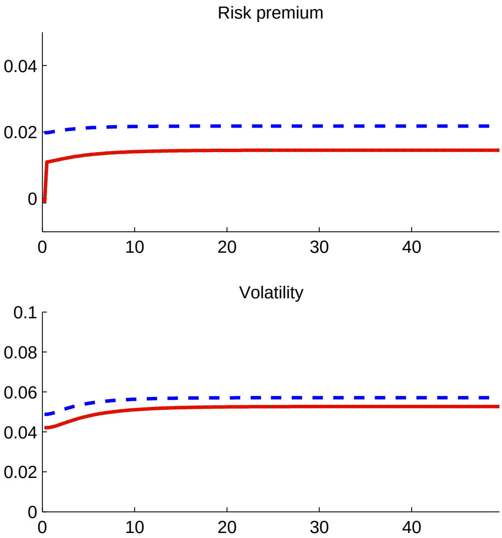


Figure: Dividend strips ($d_t = 2c_t$) in Gourio (2012)

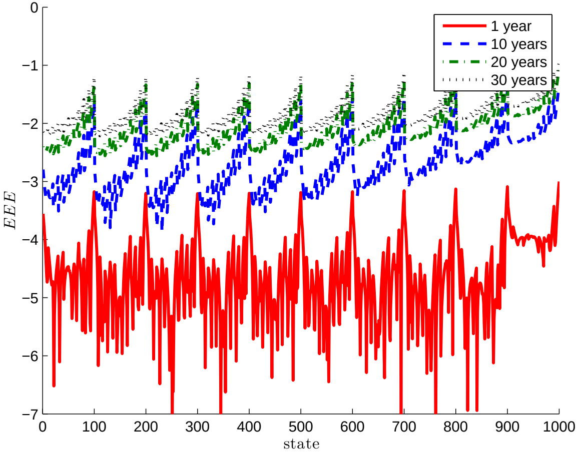


Figure: Euler Equation Errors in Gourio (2012). Projected solution

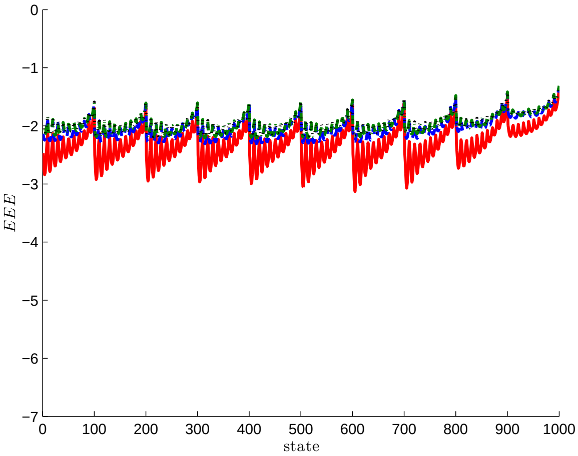


Figure: Euler Equation Errors in Gourio (2012). Affine approximation

Generalized affine approximation

A (ad-hoc) difference remains

Generalized affine:

$$z_{t+1} - E_t z_{t+1} = \widetilde{\sigma}_z(z_t)\varepsilon_{t+1}$$

Linear perturbation around the risky steady state:

$$z_{t+1} - E_t z_{t+1} = \left(\widetilde{\sigma}_z(\widetilde{z}) + [I \otimes (z_t - \widetilde{z})]\widetilde{\sigma}'_z(\widetilde{z})\right)\varepsilon_{t+1}$$

Approximation can cause spurious dynamics, especially in terms of asset pricing (bad realizations with low $\mathbb{P}$ mass can have high $\mathbb{Q}$ mass).

Generalized affine approximation

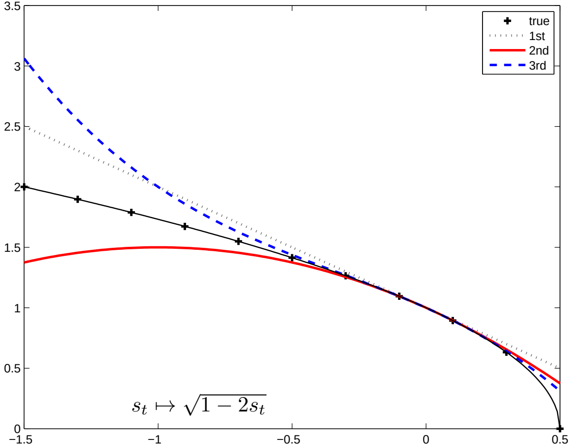


Figure: Perturbations of the map $\sqrt{1-2s_t}$ around $s_t = 0$

How to solve the nonlinear matrix equation?

Solution $[\widetilde{y}, \widetilde{z}, \widetilde{\Psi}]$ to system:

$$0 = f(\widetilde{y}, \widetilde{z}) + f_3 \widetilde{y} + f_4 \widetilde{z} + \widetilde{\mathcal{V}}(\widetilde{z}), \quad \widetilde{z} = g(\widetilde{y}, \widetilde{z})$$
$$0 = \widetilde{f}_1 \widetilde{\Psi} + \widetilde{f}_2 + (f_3 \widetilde{\Psi} + f_4)(\widetilde{g}_1 \widetilde{\Psi} + \widetilde{g}_2) + \widetilde{\mathcal{V}}_1(\widetilde{z})$$

$\widetilde{\mathcal{V}}(z_t) = \kappa[(f_3 \widetilde{\Psi} + f_4)\widetilde{\sigma}_z(z_t); z_t]$, with $\widetilde{\sigma}_z(z_t) \doteq [I_{n_z} - \lambda(z_t)\widetilde{\Psi}]^{-1} \sigma(z_t)$.

How to solve the nonlinear matrix equation?

Solution $[\widetilde{y}, \widetilde{z}, \widetilde{\Psi}]$ to system:

$$0 = f(\widetilde{y}, \widetilde{z}) + f_3 \widetilde{y} + f_4 \widetilde{z} + q \widetilde{\mathcal{V}}(\widetilde{z}), \quad \widetilde{z} = g(\widetilde{y}, \widetilde{z})$$
$$0 = \widetilde{f}_1 \widetilde{\Psi} + \widetilde{f}_2 + (f_3 \widetilde{\Psi} + f_4)(\widetilde{g}_1 \widetilde{\Psi} + \widetilde{g}_2) + q \widetilde{\mathcal{V}}_1(\widetilde{z})$$

$\widetilde{\mathcal{V}}(z_t) = \kappa[(f_3 \widetilde{\Psi} + f_4)\widetilde{\sigma}_z(z_t); z_t]$, with $\widetilde{\sigma}_z(z_t) \doteq [I_{n_z} - \lambda(z_t)\widetilde{\Psi}]^{-1}\sigma(z_t)$.

$q = 0$: linear approximation around the deterministic steady state
$q = 1$: linear approximation around the risky steady state

A simple continuation algorithm Start from the standard (i.e., saddle-path stable) linear solution at $q = 0$ and proceed sequentially until $q = 1$ with the outcome of the previous step as the initial guess for the next.

Approximate equilibrium asset pricing

Price claims in zero net supply on dividend process d_t by no arbitrage.

$$\begin{bmatrix} \widehat{z}_{t+1} \\ \Delta d_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \mu_d \end{bmatrix} + \begin{bmatrix} A \\ C \end{bmatrix} \widehat{z}_t + \begin{bmatrix} B(\widehat{z}_t) \\ D(\widehat{z}_t) \end{bmatrix} \varepsilon_{t+1}$$

$$\kappa[\alpha(z_t); z_t] \doteq \ln E_t^{\mathbb{P}}[e^{\alpha(z_t)'\varepsilon_{t+1}}], \;\; \alpha : \mathbb{R}^{n_z} \to \mathbb{R}^{n_\varepsilon}$$

with coefficients $\mu_d \in \mathbb{R}$, $A \in \mathbb{R}^{n_z \times n_z}$ and $C' \in \mathbb{R}^{n_z}$. $\varepsilon_t \overset{\mathbb{P}}{\sim} MDS$.

Stochastic discount factor:

$$m_{t+1} = -r(\widehat{z}_t) - \kappa[-\gamma(\widehat{z}_t)'; z_t] - \gamma(\widehat{z}_t)'\varepsilon_{t+1}$$

Approximate equilibrium asset pricing

nth cashflow strip yield, $y_{d,t}^{(n)} \doteq -\frac{1}{n}\ln(P_{d,t}^{(n)}/D_t)$, $P_{d,t}^{(n)} = E_t^{\mathbb{P}}(M_{t,t+n}D_{t+n})$:

$$y_{d,t}^{(n)} = -\frac{1}{n}A^{(n)} - \frac{1}{n}B_z^{(n)}\widehat{z}_t$$

with coefficients determined by the matrix difference equations

$$A^{(n)} = A^{(n-1)} + \mu_d - r(0) - \kappa[-\gamma(0)'; \widetilde{z}] + \kappa[-\gamma(0)' + V_{n-1}(0)'; \widetilde{z}]$$

$$B_z^{(n)} = B_z^{(n-1)}A + C - r_1(0) - \kappa_2[-\gamma(0)'; \widetilde{z}] + \kappa_2[-\gamma(0)' + V_{n-1}(0)'; \widetilde{z}] \\ + \kappa_1[-\gamma(0)'; \widetilde{z}]\gamma_1(0) + \kappa_1[-\gamma(0)' + V_{n-1}(0)'; \widetilde{z}][-\gamma_1(0) + V_{1,n-1}(0)]$$

with boundary condition $[A^{(0)}; B_z^{(0)}] = 0$.

$V_n(\widehat{z}_t)' \doteq D(\widehat{z}_t) + B_z^{(n)}B(\widehat{z}_t)$ controls loading on shocks of unexpected component of nth holding-period log return.

Approximate equilibrium asset pricing

Holding-period risk premium $r_{d,t+1}^{e,(n)} \doteq p_{0,t}^{(1)} + p_{d,t+1}^{(n-1)} - p_{d,t}^{(n)}$ commanded by the n-period ahead cashflow strip is:

$$\ln E_t^{\mathbb{P}} R_{d,t+1}^{e,(n)} = \kappa[-\gamma(\widehat{z}_t)'; z_t] + \kappa[V_{n-1}(\widehat{z}_t)'; z_t] - \kappa[-\gamma(\widehat{z}_t)' + V_{n-1}(\widehat{z}_t)'; z_t]$$

Coincides with negative coentropy $-\mathcal{C}_t(M_{t+1}, P_{d,t+1}^{(n-1)})$ under the exponential-affine stochastic discount factor and cashflow strip price.

Coentropy of two random variables (Hansen 2012, Backus et al. 2016):

$$\mathcal{C}_t(e^{x_{t+1}}, e^{y_{t+1}}) \doteq \mathcal{V}_t(e^{x_{t+1}+y_{t+1}}) - \mathcal{V}_t(e^{x_{t+1}}) - \mathcal{V}_t(e^{y_{t+1}})$$

Approximate equilibrium asset pricing

Vector process $[z; \Delta d]$ has approximate dynamics under risk-neutral probability $\mathbb{Q}$ with ccgf:

$$\ln E_t^{\mathbb{Q}}[e^{u_z'\widehat{z}_{t+1}+u_d\Delta d_{t+1}}] = \ln E_t^{\mathbb{P}}[e^{u_z'\widehat{z}_{t+1}+u_d\Delta d_{t+1}}] \\ + \kappa[-\gamma(\widehat{z}_t)' + u'\Sigma(\widehat{z}_t); z_t] - \kappa[-\gamma(\widehat{z}_t)'; z_t] - \kappa[u'\Sigma(\widehat{z}_t); z_t]$$

for $u = [u_z; u_d] \in \mathbb{R}^{n_z+1}$.

Approximate diagnostic decompositions

Hansen-Scheinkman (2009) decomposition: $M_{t+1} = M^T_{t+1} M^P_{t+1}$.
Use solution $[\delta; f]$ to the eigenfunction problem:

$$E^{\mathbb{P}}_t [M_{t+1} f(\widehat{z}_{t+1})] = \delta f(\widehat{z}_t)$$

for some function f and scalar $\delta \in \mathbb{R}$; then $M^T_{t+1} = \delta f(\widehat{z}_t)/f(\widehat{z}_{t+1})$ and $M^P_{t+1} = M_{t+1} f(\widehat{z}_{t+1})/\delta f(\widehat{z}_t)$.

Approximate generalized affine solution $\delta \in \mathbb{R}$ and $f(\widehat{z}_t) = e^{u_z' \widehat{z}_t}$ is:

$$\begin{aligned} 0 &= \ln(\delta) - r(0) + \kappa[-\gamma(0)' + V(0)'; \widetilde{z}] - \kappa[-\gamma(0)'; \widetilde{z}] \\ u_z' &= u_z' A \widehat{z}_t - r_1(0)\widehat{z}_t - \kappa_2[-\gamma(0); \widetilde{z}] + \kappa_2[-\gamma(0) + V(0); \widetilde{z}] \\ &\quad + \kappa_1[\gamma(0)'; \widetilde{z}]\gamma_1(0) + \kappa_1[-\gamma(0)' + V(0)'; \widetilde{z}][-\gamma(0)' + V(0)'] \end{aligned}$$

where $V(\widehat{z}_t)' \doteq u_z' B(\widehat{z}_t)$.

Related literature

Loglinear-lognormal approximations Campbell (1993), Jermann (1998), Lettau-Uhlig (2000), Lettau (2003), Uhlig (2007), Bekaert-Cho-Moreno (2010), Dew-Becker (2014), Malkhozov (2014), among many others.

Perturbations around a stochastic steady state Juillard (2011), Coeurdacier-Rey-Winant (2011), Meyer-Gohde (2016).