Entropy-based approximations of DSGE models: A unified theory of risk-adjusted linearizations

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Intro

Variation in risk sensitivity is key in many dynamic equilibrium models (time-varying risk aversion, stochastic volatility, disaster risk, ...)

How to solve for equilibrium quantities and asset prices?

Conventional linear perturbations are easy to compute, provide intuition, and have precise mathematical foundation but do not capture risk premia ⇒ asset pricing and welfare costs of fluctuations cannot be studied.
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Global solutions are accurate but computationally intensive and offer limited analytic insight.
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Conventional higher-order perturbations get analytically intractable fast; non-analytic functions; deterministic SS may be invalid expansion point.
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Finance – At least since the 1990s (Hansen, Campbell, Jermann, ...): risk corrections based on ad-hoc loglinear-lognormal/affine approximations.
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Macro – Last decade: linear perturbations around risky steady state (Coeurdacier-Rey-Winant, Juillard, Meyer-Gohde).
This paper

Affine approximations used heuristically in finance can be generalized to:

- Handle non-Gaussian shocks
- Acquire mathematical foundation
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- Handle non-Gaussian shocks
- Acquire mathematical foundation $\Rightarrow$ equivalence with linear perturbation around the risky steady state (SS).

Risky SS: ex-post shocks are 0 but agents don’t know that ex ante (Juillard, 2011; Coeurdacier et al., 2011).

Deterministic SS: ex-post shocks are 0 and agents know that ex ante.
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Affine approximations used heuristically in finance can be generalized to:

- Handle non-Gaussian shocks
- Acquire mathematical foundation \( \Rightarrow \) equivalence with linear perturbation around the risky steady state (SS).
  \( \Rightarrow \) closed-form formulas for equilibrium coefficients
  \( \Rightarrow \) counterpart to Blanchard-Kahn conditions

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Deterministic SS: ex-post shocks are 0 and agents know that ex ante.
Example 1: Campbell-Cochrane in an endowment economy

Suppose the economy is described by:

\[ 0 = \ln E_t e^{m_{t+1} + r_t} \]
\[ m_{t+1} = \ln(\beta) - \gamma \Delta c_{t+1} - \gamma \Delta s_{t+1} \]
\[ s_{t+1} = \phi s_t + \lambda(s_t)\sigma\varepsilon_{t+1} \]
\[ c_{t+1} = c_t + \sigma\varepsilon_{t+1} \]

with \( \varepsilon_t \sim Niid(0, 1) \).
Example 1: Campbell-Cochrane in an endowment economy

Suppose the economy is described by:

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Example 1: Affine approximation

We conjecture \( r_t = r + \psi_{rs} s_t \):

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$$= \ln \beta + \gamma (1 - \phi)s_t + r + \psi_{rs}s_t + \frac{\gamma^2 [1 + \lambda(s_t)]^2 \sigma^2}{2}$$
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$$\approx \ln \beta + \frac{\gamma^2 [1 + \lambda(0)]^2 \sigma^2}{2} + r + \gamma (1 - \phi) s_t + \psi_{rs} s_t + \frac{\gamma^2 [1 + \lambda(0)] \lambda'(0) \sigma^2}{2} s_t$$
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and solve for unknown coefficients $r$, $\psi_{rc}$ and $\psi_{rs}$:

$$r = -\ln \beta - \frac{\gamma^2 [1 + \lambda(0)]^2 \sigma^2}{2}$$  

precautionary saving

$$\psi_{rs} = -\gamma (1 - \phi) - \gamma [1 + \lambda(0)] \lambda'(0) \sigma^2$$  

intertemporal substitution  

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$$\psi_{rs} = -\frac{\gamma(1 - \phi)}{\gamma[1 + \lambda(0)]\lambda'(0)\sigma^2}$$

intertemporal substitution

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Let’s generalize this line of reasoning...

Let the dynamic system:

\[ 0 = \ln E_t \exp[h(y_t, z_t) + f_3 y_{t+1} + f_4 z_{t+1}] \]
\[ z_{t+1} = g(y_t, z_t) + \lambda(z_t)(E_{t+1} - E_t)y_{t+1} + \sigma(z_t)\varepsilon_{t+1} \]

where \( \varepsilon_t \sim MDS \), with conditional cumulant generating function

\[ \kappa[\alpha(z_t); z_t] = \ln E_t \exp[\alpha(z_t)'\varepsilon_{t+1}] \]
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Let the dynamic system:

\[ 0 = h(y_t, z_t) + f_3 E_t y_{t+1} + f_4 E_t z_{t+1} + \mathcal{V}_t[\exp(f_3 y_{t+1} + f_4 z_{t+1})] \]

\[ z_{t+1} = g(y_t, z_t) + \lambda(z_t)(E_{t+1} - E_t)y_{t+1} + \sigma(z_t)\varepsilon_{t+1} \]

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\]

where \(\varepsilon_t \sim MDS\), with conditional cumulant generating function

\[
\kappa[\alpha(z_t); z_t] = \ln E_t \exp[\alpha(z_t)'\varepsilon_{t+1}]
\]

We define the conditional, relative entropy of a random variable \(x\) by

\[
\mathcal{V}_t[\exp(x_{t+1})] = \ln E_t \exp(x_{t+1}) - E_t x_{t+1}
\]

It’s a non-negative measure of dispersion. If \(x \sim N(0, \sigma^2)\), then \(\mathcal{V}(e^x) = \frac{\sigma^2}{2}\)
Let’s generalize this line of reasoning...

Let the dynamic system:

\[
0 = h(y_t, z_t) + f_3 E_t y_{t+1} + f_4 E_t z_{t+1} + \mathcal{V}_t[\exp(f_3 y_{t+1} + f_4 z_{t+1})]
\]
\[
z_{t+1} = g(y_t, z_t) + \lambda(z_t)(E_{t+1} - E_t)y_{t+1} + \sigma(z_t)\varepsilon_{t+1}
\]

where \(\varepsilon_t \sim MDS\), with conditional cumulant generating function

\[
\kappa[\alpha(z_t); z_t] = \ln E_t \exp[\alpha(z_t)'\varepsilon_{t+1}]
\]

We are looking for an affine solution with unknowns \([\tilde{y}; \tilde{z}; \tilde{\Psi}]\):

\[
y_t = \tilde{y} + \tilde{\Psi}(z_t - \tilde{z})
\]

hence:

\[
z_{t+1} = g(y_t, z_t) + \tilde{\sigma}_z(z_t)\varepsilon_{t+1}, \quad \tilde{\sigma}_z(z_t) = \left[\ln z - \lambda(z_t)\tilde{\Psi}\right]^{-1}\sigma(z_t)
\]

\[
\tilde{\mathcal{V}}(z_t) = \mathcal{V}_t[\exp((f_3\tilde{\Psi} + f_4)z_{t+1})] = \kappa[(f_3\tilde{\Psi} + f_4)\tilde{\sigma}_z(z_t); z_t]
\]
Let’s generalize this line of reasoning...

Given system:

\[
0 = \ln E_t \exp[h(y_t, z_t) + f_3 y_{t+1} + f_4 z_{t+1}]
\]
\[
z_{t+1} = g(y_t, z_t) + \lambda(z_t)(E_{t+1} - E_t)y_{t+1} + \sigma(z_t)\varepsilon_{t+1}
\]

the affine approximate solution is:

\[
y_t = \tilde{y} + \tilde{\Psi}(z_t - \tilde{z})
\]
\[
z_{t+1} = \tilde{z} + \tilde{g}_1(y_t - \tilde{y}) + \tilde{g}_2(z_t - \tilde{z}) + [I_{n_z} - \lambda(z_t)\tilde{\Psi}]^{-1}\sigma(z_t)\varepsilon_{t+1}
\]

where the unknowns \([\tilde{y}, \tilde{z}, \tilde{\Psi}]\) solve the system of equations:

\[
\tilde{z} = g(\tilde{y}, \tilde{z})
\]
\[
0 = h(\tilde{y}, \tilde{z}) + f_3 \tilde{y} + f_4 \tilde{z} + \tilde{\mathcal{V}}(\tilde{z})
\]
\[
0 = \tilde{f}_1 \tilde{\Psi} + \tilde{f}_2 + (f_3 \tilde{\Psi} + f_4)(\tilde{g}_1 \tilde{\Psi} + \tilde{g}_2) + \tilde{\mathcal{V}}_1(\tilde{z})
\]

with \(\tilde{\mathcal{V}}(z_t) \doteq \kappa \left[ (f_3 \tilde{\Psi} + f_4)[I_{n_z} - \lambda(z_t)\tilde{\Psi}]^{-1}\sigma(z_t); z_t \right] \).
Affine approximation: Formal derivation


Proposition

The point \((y_t, z_t, q) = (\tilde{y}, \tilde{z}, 1)\) is a saddle point if and only if the generalized eigenvalues

\[
\alpha(\Gamma, \Xi) \doteq \{ \alpha \in \mathbb{C} : \det(\Gamma \alpha - \Xi) = 0 \} = \{ \alpha_i, i = 1, \ldots, n_y + n_z \}
\]

of the square matrices:

\[
\Gamma \doteq \begin{bmatrix} f_4 & f_3 \\ f_3 \end{bmatrix}, \quad \Xi \doteq \begin{bmatrix} -f_2(\tilde{y}, \tilde{z}) - \tilde{V}_1(\tilde{z}) & -f_1(\tilde{y}, \tilde{z}) \\ g_2(\tilde{y}, \tilde{z}) & g_1(\tilde{y}, \tilde{z}) \end{bmatrix}
\]

are such that there are \(n_z\) generalized eigenvalues with modulus within the unit circle and \(n_y\) with modulus larger than unity.
Affine approximation: Formal derivation

Consider the parametrized family of system (1):

\[ 0 = E_t x_{t+1}^q + \nu_t^q, \quad \nu_t^q = \mathcal{V}_t[\exp(x_{t+1}^q)] \]

\[ z_{t+1}^q = g[y(z_t, q), z_t] + \lambda(z_t)(E_{t+1} - E_t)y[z(z_t, q, q\varepsilon_{t+1}), q] + q\sigma(z_t)\varepsilon_{t+1} \]

\[ x_{t+1}^q = h[y(z_t, q), z_t] + f_3y[z(z_t, q, q\varepsilon_{t+1}), q] + f_4z(z_t, q, q\varepsilon_{t+1}) \]

where the perturbation scalar \( q \in [0, 1] \) indexes the system’s sensitivity to shocks, so that under \( q = 1 \) the dynamics coincide with (1).

Let \( \bar{z} = z(\bar{z}, 1, 0) \) a risky steady state.

**Proposition**

Suppose that a risky steady state \((z_t, q) = (\bar{z}, 1)\) is a saddle point. Then, the affine approximate solution locally unique and stable, and the solution \([\tilde{y}, \tilde{z}, \tilde{\Psi}]\) to system (2) coincides with the coefficients from a linear perturbation around the risky steady state:

\[ \tilde{y} = y(\bar{z}, 1), \quad \tilde{z} = \bar{z}, \quad \tilde{\Psi} = y_1(\bar{z}, 1) \]
Affine approximations are not nested in conventional perturbations around the deterministic steady state. We can at most reconstruct implicit functions $y(0, q)$ and $y_1(0, q)$ using output from $N$-order perturbations around $(z_t, q) = (0, 0)$ as:

$$y(0, q) = \lim_{N \to \infty} \sum_{i=1}^{N} \frac{1}{i!} \frac{\partial^i y(0, q)}{\partial q^i} \bigg|_{q=0} q^i, \quad y_1(0, q) = \lim_{N \to \infty} \sum_{i=0}^{N} \frac{1}{i!} \frac{\partial^i y_1(0, q)}{\partial q^i} \bigg|_{q=0} q^i$$

as long as $y(0, q)$ and $y_1(0, q)$ have convergent Taylor series at $q = 0$ with a sufficiently large radius of convergence. Functions of interest: $y(\bar{z}, 1)$ and $y_1(\bar{z}, 1)$.

**Proposition**

If $y(\bar{z}, 1) \neq y(0, 1)$ or $y_1(\bar{z}, 1) \neq y_1(0, 1)$, then risky steady state perturbations are not nested in deterministic steady state perturbations of arbitrary order $N$. 

Key examples

- Campbell–Cochrane (1999)
- Wachter (2013)
- Coeurdacier–Rey–Winant (2011)
- Jermann (1998)
- Gourio (2012)
Campbell-Cochrane (1999) endowment economy
Log price-consumption ratio of wealth portfolio solves the forward-looking difference equation:

\[ e^{pc_t} = E_t e^{\ln(\beta) + (1-\gamma)\Delta c_{t+1} - \gamma \Delta s_{t+1}} + E_t e^{\ln(\beta) + (1-\gamma)\Delta c_{t+1} - \gamma \Delta s_{t+1}} + pc_{t+1} \]

\[ = E_t e^{\ln(\beta) + (1-\gamma)\Delta c_{t+1} - \gamma \Delta s_{t+1}} + \ln(1 + e^{pc_{t+1}}) \]

When combined with a terminal condition that rules out bubbles, the forward-looking difference equation unwinds as:

\[ pc_t = \ln \left( \sum_{n=1}^{\infty} e^{pc_t^{(n)}} \right), \quad pc_t^{(n)} = \ln E_t e^{\ln(\beta) + (1-\gamma)\Delta c_{t+1} - \gamma \Delta s_{t+1}} + pc_t^{(n-1)} \]

where \( pc_t^{(n)} \) describes the log price-consumption ratio of the \( n \)th consumption strip, with \( pc_t^{(0)} = 0 \).
Campbell-Cochrane (1999): Pricing wealth

Step 1. Split into certainty-equivalent and entropy terms:

\[
0 = \ln(\beta) + (1 - \gamma)E_t \Delta c_{t+1} - \gamma E_t \Delta s_{t+1} + E_t p c_{t+1}^{(n-1)} - p c_t^{(n)} \\
+ \mathcal{V}_t \left( e^{(1 - \gamma) \Delta c_{t+1} - \gamma \Delta s_{t+1} + pc_{t+1}^{(n-1)}} \right)
\]
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\[ 0 = \ln(\beta) + (1 - \gamma)E_t \Delta c_{t+1} - \gamma E_t \Delta s_{t+1} + E_t pc_{t+1}^{(n-1)} - pc_t^{(n)} \]

\[ + \mathcal{V}_t \left( e^{(1-\gamma)\Delta c_{t+1} - \gamma \Delta s_{t+1} + pc_{t+1}^{(n-1)}} \right) \]

Step 2. Guess \( pc_t^{(n)} = \tilde{p}c^{(n)} + \tilde{\psi}^{(n)} \hat{s}_t \), hence:

\[ \mathcal{V}_t \left( e^{(1-\gamma)\Delta c_{t+1} - \gamma \Delta s_{t+1} + pc_{t+1}^{(n-1)}} \right) = \left( 1 - \gamma [1 + \Lambda(\hat{s}_t)] + \tilde{\psi}^{(n-1)} \Lambda(\hat{s}_t) \right)^2 \frac{\sigma^2}{2} \]
Campaign-Cochrane (1999): Pricing wealth

Step 3.

\[ 0 = \ln(\beta) + (1 - \gamma)E_t \Delta c_{t+1} - \gamma E_t \Delta s_{t+1} + E_t pc_{t+1}^{(n-1)} - pc_t^{(n)} \]

\[ + \left(1 - \gamma [1 + \Lambda(\hat{s}_t)] + \tilde{\psi}^{(n)} \Lambda(\hat{s}_t)\right)^2 \sigma^2 \frac{\hat{s}_t}{2} \]
Campbell-Cochrane (1999): Pricing wealth

Step 3. Linearize:

\[
0 = \ln(\beta) + (1 - \gamma)E_t \Delta c_{t+1} - \gamma E_t \Delta s_{t+1} + E_t pc_{t+1}^{(n-1)} - pc_t^{(n)}
+ \left(1 - \gamma [1 + \Lambda(\hat{s}_t)] + \tilde{\psi}^{(n)}\Lambda(\hat{s}_t)\right)^2 \sigma^2 \frac{2}{2}
\]

\[
\approx \tilde{p}c^{(n-1)} - \tilde{p}c^{(n)} + \ln(\beta e^{(1-\gamma)\mu}) + \left(1 - \gamma [1 + \Lambda(0)] + \tilde{\psi}^{(n-1)}\Lambda(0)\right)^2 \frac{\sigma^2}{2}
+ \tilde{\psi}^{(n-1)} \phi \hat{s}_t - \tilde{\psi}^{(n)}\hat{s}_t + \gamma (1 - \phi)\hat{s}_t
+ \left(1 - \gamma [1 + \Lambda(0)] + \tilde{\psi}^{(n-1)}\Lambda(0)\right) \left(\tilde{\psi}^{(n-1)} - \gamma \Lambda_1(0)\sigma^2\hat{s}_t\right)
\]
Campbell-Cochrane (1999): Pricing wealth

Step 4. Identify:

\[ \tilde{pc}^{(n)} = \tilde{pc}^{(n-1)} + \ln(\beta e^{(1-\gamma)\mu}) + \left(1 - \gamma[1 + \Lambda(0)] + \tilde{\psi}^{(n-1)}\Lambda(0)\right)^2 \frac{\sigma^2}{2} \]

\[ \tilde{\psi}^{(n)} = \tilde{\psi}^{(n-1)}\phi + \gamma(1 - \phi) + \left(1 - \gamma[1 + \Lambda(0)] + \tilde{\psi}^{(n-1)}\Lambda(0)\right) (\tilde{\psi}^{(n-1)} - \gamma)\Lambda_1(0)\sigma^2 \]

with boundary condition \( \tilde{pc}^{(0)} = \tilde{\psi}^{(0)} = 0 \). The approximate solution is:

\[ pc_t = \ln \left( \sum_{n=1}^{\infty} e^{p^c_{t}^{(n)}} \right), \quad p^c_{t}^{(n)} = \tilde{pc}^{(n)} + \tilde{\psi}^{(n)}s_t \]
Campbell-Cochrane (1999): Pricing wealth

Compare to a conventional third-order perturbation:

\[
\hat{s}_{t+1} = \phi \hat{s}_t + \Lambda(0)\sigma \varepsilon_{t+1} + \frac{1}{2} \Lambda_{11}(0)\hat{s}_t^2 \sigma \varepsilon_{t+1}
\]

where:

\[
\bar{\psi}^{(n)}_1 = \gamma(1 - \phi^n)
\]

\[
\bar{\psi}^{(n)}_2 = \bar{\psi}^{(n-1)}_2 + \left(1 - \gamma[1 + \Lambda(0)] + \bar{\psi}^{(n-1)}_1 \Lambda(0)\right)^2 \frac{\sigma^2}{2}
\]

\[
\bar{\psi}^{(n)}_3 = \bar{\psi}^{(n-1)}_3 \phi + \left(1 - \gamma[1 + \Lambda(0)] + \bar{\psi}^{(n-1)}_1 \Lambda(0)\right) (\bar{\psi}^{(n-1)}_1 - \gamma)\Lambda_1(0)\sigma^2
\]

with boundary conditions \( \bar{\psi}^{(0)}_2 = \bar{\psi}^{(0)}_3 = 0 \).
Applications

\( P_{c,t}/C_t \)

- Global (100-point grid)
- Global (20-point grid)
- Affine
- Affine 2
- Affine 3
- 3rd order

\( S_t \)

Pier Lopez (BdF)
Figure: Consumption strips in Campbell-Cochrane (1999)
Figure: Dividend strips in Campbell-Cochrane (1999)
Figure: Real bonds in Campbell-Cochrane (1999)
Wachter (2013) endowment economy with recursive preferences and variable disaster risk.
Coeurdacier-Rey-Winant (2011) small open economy with intertemporal consumption choice.
Jermann (1998) production economy with capital adjustment costs and Campbell-Cochrane habits
The graph shows the relationship between $I_t/K_t$ and $S_t$ for different levels of $K_t/A_t$: low, mid, and high. The curves indicate how the ratio of investment to capital evolves with respect to technology, with distinct patterns for each level of capital to asset ratio.
Figure: Consumption strips in Jermann (1998) with nonlinear habits
Figure: Dividend strips in Jermann (1998) with nonlinear habits
**Figure**: Real bonds in Jermann (1998) with nonlinear habits.
Figure: EEE in Jermann (1998) with CC habits. Projected solution
Figure: EEE in Jermann (1998) with nonlinear habits. Affine approximation
Figure: Dividend strips in Lopez et al. (2015)
Figure: Nominal bonds in Lopez et al. (2015)
Figure: Real bonds in Lopez et al. (2015)
Figure: Euler Equation Errors in Lopez et al. (2015). Projected solution
Figure: Euler Equation Errors in Lopez et al. (2015). Affine approximation
Gourio (2012) production economy with recursive preferences and variable disaster risk
Figure: Real bonds ($d_t = 0$) in Gourio (2012)
Figure: Consumption strips \((d_t = c_t)\) in Gourio (2012)
Figure: Dividend strips \( (d_t = 2c_t) \) in Gourio (2012)
Figure: Euler Equation Errors in Gourio (2012). Projected solution
Figure: Euler Equation Errors in Gourio (2012). Affine approximation
Generalized affine approximation

A (ad-hoc) difference remains

Generalized affine:

\[ z_{t+1} - E_t z_{t+1} = \tilde{\sigma}_z(z_t) \varepsilon_{t+1} \]

Linear perturbation around the risky steady state:

\[ z_{t+1} - E_t z_{t+1} = (\tilde{\sigma}_z(\tilde{z}) + [I \otimes (z_t - \tilde{z})] \tilde{\sigma}'(\tilde{z})) \varepsilon_{t+1} \]

Approximation can cause spurious dynamics, especially in terms of asset pricing (bad realizations with low \( \mathbb{P} \) mass can have high \( \mathbb{Q} \) mass).
Generalized affine approximation

\[ s_t \mapsto \sqrt{1 - 2s_t} \]

**Figure:** Perturbations of the map \( \sqrt{1 - 2s_t} \) around \( s_t = 0 \)
How to solve the nonlinear matrix equation?

Solution $[\tilde{y}, \tilde{z}, \tilde{\Psi}]$ to system:

$$0 = f(\tilde{y}, \tilde{z}) + f_3\tilde{y} + f_4\tilde{z} + \tilde{\mathcal{V}}(\tilde{z}), \quad \tilde{z} = g(\tilde{y}, \tilde{z})$$

$$0 = \tilde{f}_1\tilde{\Psi} + \tilde{f}_2 + (f_3\tilde{\Psi} + f_4)(\tilde{g}_1\tilde{\Psi} + \tilde{g}_2) + \tilde{\mathcal{V}}_1(\tilde{z})$$

$$\tilde{\mathcal{V}}(z_t) = \kappa[(f_3\tilde{\Psi} + f_4)\tilde{\sigma}_z(z_t); z_t], \text{ with } \tilde{\sigma}_z(z_t) \doteq [I_{n_z} - \lambda(z_t)\tilde{\Psi}]^{-1}\sigma(z_t).$$
How to solve the nonlinear matrix equation?

Solution $[\tilde{y}, \tilde{z}, \tilde{\Psi}]$ to system:

\[
0 = f(\tilde{y}, \tilde{z}) + f_3\tilde{y} + f_4\tilde{z} + q\tilde{\mathcal{V}}(\tilde{z}), \quad \tilde{z} = g(\tilde{y}, \tilde{z})
\]
\[
0 = \tilde{f}_1\tilde{\Psi} + \tilde{f}_2 + (f_3\tilde{\Psi} + f_4)(\tilde{g}_1\tilde{\Psi} + \tilde{g}_2) + q\tilde{\mathcal{V}}_1(\tilde{z})
\]

\[
\tilde{\mathcal{V}}(z_t) = \kappa[(f_3\tilde{\Psi} + f_4)\tilde{\sigma}_z(z_t); z_t], \text{ with } \tilde{\sigma}_z(z_t) \triangleq [I_{n_z} - \lambda(z_t)\tilde{\Psi}]^{-1}\sigma(z_t).
\]

$q = 0$: linear approximation around the deterministic steady state
$q = 1$: linear approximation around the risky steady state

**A simple continuation algorithm** Start from the standard (i.e., saddle-path stable) linear solution at $q = 0$ and proceed sequentially until $q = 1$ with the outcome of the previous step as the initial guess for the next.
Approximate equilibrium asset pricing

Price claims in zero net supply on dividend process $d_t$ by no arbitrage.

\[
\begin{bmatrix}
\hat{z}_{t+1} \\
\Delta d_{t+1}
\end{bmatrix}
= \begin{bmatrix} 0 \\ \mu_d \end{bmatrix}
+ \begin{bmatrix} A \\ C \end{bmatrix} \hat{z}_t
+ \begin{bmatrix} B(\hat{z}_t) \\ D(\hat{z}_t) \end{bmatrix} \varepsilon_{t+1}
\]

\[
\kappa[\alpha(z_t); z_t] \triangleq \ln E_t^P [e^{\alpha(z_t)'} \varepsilon_{t+1}], \quad \alpha : \mathbb{R}^{n_z} \to \mathbb{R}^{n_\varepsilon}
\]

with coefficients $\mu_d \in \mathbb{R}$, $A \in \mathbb{R}^{n_z \times n_z}$ and $C' \in \mathbb{R}^{n_z}$. $\varepsilon_t \overset{\text{P}}{\sim} \text{MDS}$.

Stochastic discount factor:

\[
m_{t+1} = -r(\hat{z}_t) - \kappa[-\gamma(\hat{z}_t)'; z_t] - \gamma(\hat{z}_t)' \varepsilon_{t+1}
\]
Approximate equilibrium asset pricing

nth cashflow strip yield, \( y_{d,t}^{(n)} = -\frac{1}{n} \ln(P_{d,t}^{(n)}/D_t) \), \( P_{d,t}^{(n)} = E_t^{\text{IP}}(M_{t,t+n}D_{t+n}) \):

\[
y_{d,t}^{(n)} = -\frac{1}{n} A^{(n)} - \frac{1}{n} B_z^{(n)} \hat{z}_t
\]

with coefficients determined by the matrix difference equations

\[
A^{(n)} = A^{(n-1)} + \mu_d - r(0) - \kappa[-\gamma(0)'; \hat{z}] + \kappa[-\gamma(0)' + V_{n-1}(0)'; \hat{z}]
\]

\[
B_z^{(n)} = B_z^{(n-1)} A + C - r_1(0) - \kappa_2[-\gamma(0)'; \hat{z}] + \kappa_2[-\gamma(0)' + V_{n-1}(0)'; \hat{z}]
\]

\[
+ \kappa_1[-\gamma(0)'; \hat{z}] \gamma_1(0) + \kappa_1[-\gamma(0)' + V_{n-1}(0)'; \hat{z}][-\gamma_1(0) + V_{1,n-1}(0)]
\]

with boundary condition \([A^{(0)}; B_z^{(0)}] = 0\).

\( V_n(\hat{z}_t)' = D(\hat{z}_t) + B_z^{(n)} B(\hat{z}_t) \) controls loading on shocks of unexpected component of nth holding-period log return.
Approximate equilibrium asset pricing

Holding-period risk premium \( r_{d,t+1}^{e,(n)} = p_{0,t}^{(1)} + p_{d,t+1}^{(n-1)} - p_{d,t}^{(n)} \) commanded by the \( n \)-period ahead cashflow strip is:

\[
\ln E_t^P R_{d,t+1}^{e,(n)} = \kappa[-\gamma(\hat{Z}_t)' ; z_t] + \kappa[V_{n-1}(\hat{Z}_t)' ; z_t] - \kappa[-\gamma(\hat{Z}_t)' + V_{n-1}(\hat{Z}_t)' ; z_t]
\]

Coincides with negative coentropy \(-C_t(M_{t+1}, P_{d,t+1}^{(n-1)})\) under the exponential-affine stochastic discount factor and cashflow strip price.

Coentropy of two random variables (Hansen 2012, Backus et al. 2016):

\[
C_t(e^{X_{t+1}}, e^{Y_{t+1}}) \doteq V_t(e^{X_{t+1}+Y_{t+1}}) - V_t(e^{X_{t+1}}) - V_t(e^{Y_{t+1}})
\]
Approximate equilibrium asset pricing

Vector process \([z; \Delta d]\) has approximate dynamics under risk-neutral probability \(\mathbb{Q}\) with cggf:

\[
\ln E_t^\mathbb{Q}[e^{u_z'\hat{z}_{t+1} + u_d \Delta d_{t+1}}] = \ln E_t^\mathbb{P}[e^{u_z'\hat{z}_{t+1} + u_d \Delta d_{t+1}}] \\
+ \kappa [\gamma (\hat{z}_t)' + u'\Sigma (\hat{z}_t); z_t] - \kappa [\gamma (\hat{z}_t)'; z_t] - \kappa [u'\Sigma (\hat{z}_t); z_t]
\]

for \(u = [u_z; u_d] \in \mathbb{R}^{n_z+1}\).
Approximate diagnostic decompositions

Hansen-Scheinkman (2009) decomposition: \( M_{t+1} = M^T_{t+1} M^P_{t+1} \).

Use solution \([\delta; f]\) to the eigenfunction problem:

\[
E^P_t[M_{t+1} f(\hat{z}_{t+1})] = \delta f(\hat{z}_t)
\]

for some function \( f \) and scalar \( \delta \in \mathbb{R} \); then \( M^T_{t+1} = \delta f(\hat{z}_t)/f(\hat{z}_{t+1}) \) and \( M^P_{t+1} = M_{t+1} f(\hat{z}_{t+1})/\delta f(\hat{z}_t) \).

Approximate generalized affine solution \( \delta \in \mathbb{R} \) and \( f(\hat{z}_t) = e^{u'_z\hat{z}_t} \) is:

\[
0 = \ln(\delta) - r(0) + \kappa[-\gamma(0)' + V(0); \hat{z}] - \kappa[-\gamma(0)'; \hat{z}]
\]

\[
u'_z = u'_z A\hat{z}_t - r_1(0)\hat{z}_t - \kappa_2[-\gamma(0); \hat{z}] + \kappa_2[-\gamma(0) + V(0); \hat{z}]
\]
\[+ \kappa_1 [\gamma(0)'; \hat{z}] \gamma_1(0) + \kappa_1[-\gamma(0)' + V(0)'; \hat{z}] [-\gamma(0)' + V(0)'] \]

where \( V(\hat{z}_t)' = u'_z B(\hat{z}_t) \).
Related literature
