Ambiguous Business Cycles: A Quantitative Assessment*

Sumru Altug †,1, Fabrice Collard ,2, Cem Çakmaklı,3, Sujoy Mukerji ,4, and Han Özsöylev 3,5

[1] *American University of Beirut and CEPR*
[2] *University of Toulouse and CEPR*
[3] *Koç University*
[4] *Queen Mary University*
[5] *University of Oxford*

June 17, 2019

Abstract

In this paper, we examine the cyclical dynamics of a business cycle model with ambiguity averse consumers and investment irreversibility using the smooth ambiguity model of Klibanoff *et al.* (2005, 2009). The model differentiates between the sources of ambiguity and ambiguity aversion, and allows for learning about an unobserved cyclical component of TFP. Using aggregate TFP data, we find that the model has the ability to generate significant effects of ambiguity and ambiguity aversion when the impact of ambiguity is modeled as greater informativeness regarding estimates of the underlying TFP process. In our analysis, we measure such informativeness through the use of priors with greater signal-to-noise ratios which enable the agent to discriminate better between the low and high persistent processes. Viewing this as a shock to the agent's "confidence" regarding the nature of the underlying TFP process, we find that greater confidence or lower ambiguity tends to lower the overall cyclical variability of investment and hours and to increase the responsiveness of consumption and output by lessening precautionary saving effects. It also leads to greater co-movement in all of the key macroeconomic aggregates. In addition to examining business cycle moments, we show that ambiguity manifests itself through the risk-free rate. Specifically, the greater the endogenous distortions induced by ambiguity and ambiguity aversion, the lower is the risk-free rate in equilibrium. In contrast to our earlier studies, we make use of both aggregate and sectoral TFP data and find that sectoral TFP data allow for the ability to differentiate the impact of ambiguity through the properties of the filtered and distorted beliefs that comprise the risk-free rate. Finally, we find that standard uncertainty measures do not always correlate with ambiguity measures based on similar data.

Keywords: Ambiguity, ambiguity aversion, information and learning, investment irreversibility, Real Business Cycles.

JEL Classification: C6, D8, E2

*We are grateful for comments from Nicholas Bloom, Benjamin Born, Steve Davis, Alan Paquet, and Levent Koçkesen. An earlier version of this paper was presented at the Third Research Conference of the Macroeconomic Modelling and Model Comparison Network (MMCN), Frankfurt, 2019; the Asia-Pacific Economic Association (APEA) Conference, Los Angeles, 2018; Session on Prices, Business Fluctuations and Cycles at the International Atlantic Economic Society Meetings, Montreal, 2017; the 6^{th} Delhi Macroeconomics Workshop at the Indian Statistical Institute, Delhi, India, 2017; the Central Bank of the Republic of Turkey, 2017; and seminars at Groningen University, Koç University, Özyeğin University and American University Beirut. This research was supported by a Newton Mobility Grant from the British Academy, Grant Number NG160021.

†Correspondence to: Sumru Altug, American University of Beirut, P.O.Box 11-0236 / Riad El-Solh / Beirut 1107 2020, Lebanon , e–mail: sa287@aub.edu.lb

1 Introduction

The notion that ambiguity affects the choices of economic agents has gained currency in the recent literature. The multiple priors framework of Gilboa and Schmeidler (1989) formulated ambiguity in terms of a given set of priors over the objective distributions generating observed outcomes and ambiguity aversion in terms of the worse case distribution from these priors; see, for example, Ilut and Schneider (2014) or Bianchi *et al.* (2017). The smooth ambiguity model introduced by Klibanoff *et al.* (2005, 2009), denoted as KMM hereafter, allows for a separate treatment of ambiguity and ambiguity aversion by assuming that ambiguity arises from the problem of inferring the true probability distribution generating fluctuations in future fundamentals while ambiguity aversion is defined as a mean-preserving spreads in the distributions of expected utility values. These authors show that ambiguity aversion in agents' preferences endogenously generates "doubt and pessimism" about the external environment.

In an application of this approach, Collard *et al.* (2018) study the historical equity premium in an endowment economy framework. They find that ambiguity aversion accentuates the conditional uncertainty embodied in U.S. macroeconomic growth outcomes endogenously. After calibrating the ambiguity aversion parameter to match the risk-free rate, they are able to match the first and second conditional moments of observed return dynamics. However, their model is based on a non-stationary representation of preferences, thus preventing the solution of their model economy in the presence of production and capital accumulation. Instead, following Bansal and Yaron (2004) and Campbell (1996), they assume that consumption is some multiple of the observed dividend process and evaluate their model for a single realization of the observed consumption process.

In this paper, we adopt the smooth ambiguity preferences of Klibanoff *et al.* (2005, 2009) to examine the cyclical dynamics of a Real Business Cycle model with irreversible investment and labor augmenting technology shocks. In this framework, the shock to aggregate TFP evolves as a function of a latent variable governing its persistence. Ambiguity is introduced by assuming that agents are unsure about the distribution of the latent variable in that they cannot distinguish a process that has moderate persistence but high volatility, and one which is less volatile but highly persistent. Our approach admits a stationary representation for the decision rules of the model, which are used to characterize optimal investment and hours choices under alternative processes for TFP growth. Following the literature on non-expected utility models that allows for early versus late resolution of uncertainty, Ju and Miao (2007, 2012) extend the parametric preference frameworks that distinguish between risk aversion and intertemporal substitution to also include a role for ambiguity aversion. Jahan-Parvar and Liu (2014) and Liu and Zhang (2018) use generalized recursive smooth ambiguity model proposed by Ju and Miao (2007, 2012) to account for asset pricing phenomena in a Real Business Cycle Model augmented with adjustment costs in investment and a Markov switching process for aggregate productivity growth.[1] Bidder and Smith (2012) consider a version of the multiplier preferences that are robust to misspecification following Hansen and Sargent (2008) and assume that innovations to technology growth are subject to time-varying volatility. Ilut and Schneider (2014) develop a New Keynesian model of business cycle fluctuations within the multiple priors framework which allows for log-linear solution methods and which capture the impact of ambiguity and ambiguity aversion in a first-order manner. Nimark (2014) develops a business cycle model which combines higher order beliefs - expectations about the expectations of others - with the existence of a public signal that is more likely to be observed after unusual events. Such 'man-bites-dog' type signals increase uncertainty as well as disagreement among agents, and are able to account for periods of large changes in aggregate activity without large changes in underlying fundamentals. Gallant *et al.* (2015) use macroeconomic and financial data together with Bayesian inference to measure the size of ambiguity aversion in a Lucas-type consumption-based asset pricing model.

In these models, the specification of uncertainty/ambiguity occupies a key role, as does the determination of agents' beliefs about the latent states and their distributions. In the multiple priors framework, a loss in confidence is modeled as an increase in the size of the set of beliefs from which the worst case beliefs are drawn. Ambiguity and changes in confidence are incorporated into a business cycle model by assuming that "agents' set of beliefs, such as an innovation to productivity,

[1] For example, Liu and Zhang (2018) seek to establish the cross-correlations between equity returns (or the "variance risk premium") and variables typically affected/indicated by the business cycle in a production economy with ambiguity.

is parameterized by an interval of means centered around zero." An increase in the width of the interval is associated with a loss of confidence, especially when the worst case mean becomes worse. One problem with this approach is that ambiguity and ambiguity aversion are both determined by the size of the set of possible beliefs. This literature also links to the literature on news shocks in that changes in confidence behave like a news shock (see Beadry and Portier (2006), Jaimovich and Rebelo (2009)). However, the difference between changes in confidence and news shocks is that the latter are followed, at least on average, by a shock realization that corroborates the news, but this need not be the case for changes in confidence.

The issue of the extent of knowledge possessed by the econometrician versus agents is also plays a key role in the literature on ambiguity and/or robust decision-making (Hansen (2007)). In our model, the agent and the econometrician possess the same information and must infer the nature of the true TFP growth process based on past observations of TFP growth. Ju and Miao (2012) model the growth rate of consumption and dividends as a hidden two-state Markov switching process, where agents learn about the hidden state using past data. In our framework (see also Collard *et al.* (2018)), agents believe that the growth rate of consumption and dividends are partly driven by a common latent state, x_t, which evolves according to a first-order autoregressive process with persistence parameter, ρ and variance, σ^2. While it is assumed that there is a single persistence/volatility parameter operating throughout history, agents do not know whether variation in TFP growth is driven by a process with high persistence and low volatility, or one with lower persistence but higher volatility, and they must make inferences about the probability of one of these processes being the true process at the same as they infer the behavior of the unobserved temporary component using a Kalman filtering algorithm. In contrast to a situation with learning about an unobserved temporary component drawn from a known probability distribution, ambiguity arises from the problem of inferring the true probability distribution generating the temporary fluctuations in future fundamentals. In both cases, the rational expectations assumption that agents are endowed with more precise information than the econometrician is relaxed.

In this paper, we take a prototypical Real Business Cycle model with ambiguity averse consumers and investment irreversibility to examine the evolution of beliefs under ambiguity, information and learning and their impact on the cyclical dynamics of real variables (such as consumption, investment, hours and output). First, we show information and learning effects are key to how ambiguity and ambiguity aversion affect endogenous choices. Backus *et al.* (2015) have argued that learning together with the role of ambiguity may prove useful for generating the observed business cycle dynamics in that learning provides another source of dynamics, which have been exploited by Collard *et al.* (2018) and Collin-Dufresne *et al.* (2016) to generate significant effects on asset prices. We contribute to the recent literature with ambiguity and ambiguity aversion by providing an explicit discussion of the decision problem of the agent under productivity processes estimated by assuming increasing degrees of informativeness about the underlying unknown latent process driving temporary fluctuations in productivity growth. We may view more informative prior distributions as a measure of the agent's "confidence" regarding the nature of the underlying productivity process.[2] Specifically, we show that more precise beliefs regarding the unknown process generating aggregate productivity are, in general, associated with lower cyclical variability in investment and hours. On the other hand, the variability of output and consumption tends to increase with greater information if agents know with more certainty the nature of the latent process generating the observations, as does the difference in the consumption and output responses across the high and low persistence processes. Put differently, the precautionary saving motive becomes lessened under greater information or lower ambiguity. On the other hand, the co-movement among the different series increases with greater informativeness or less ambiguity about the underlying TFP process across all parameter configurations as well as the nature of the TFP process that is being simulated.

The nature of the underlying productivity process also has implications for the role of ambiguity and ambiguity aversion on the behavior of asset prices and in particular, the risk-free rate. As is well known, an ambiguity averse agent has "as if" beliefs which are more pessimistic relative to a Bayesian learner placed in a similar environment. We show using both a log-linear approximation of a version of the model with known persistence as well as numerical solutions of the full model with unknown persistence that the greater the endogenous distortions induced by ambiguity and ambiguity aversion, the lower is the risk-free rate in equilibrium. In contrast to earlier studies such

[2] Angeletos and La'O (2013) provide another formulation of confidence in a Real Business Cycle with incomplete information.

as Ilut and Schneider (2014) or Bidder and Smith (2012), our analysis examines both aggregate and sectoral data to understand the quantitative impact of uncertainty on prices and quantities and emphasizes the information channel in the transmission of ambiguity and ambiguity aversion. In addition to examining business cycle moments, we show that ambiguity manifests itself through the risk-free rate. In this respect, we find that sectoral TFP data allow for the ability to differentiate the impact of ambiguity through the properties of the filtered and distorted beliefs that comprise the risk-free rate. Finally, we find that standard uncertainty measures do not always correlate with ambiguity measures based on similar data.

The remainder of this paper is organized as follows. Section 2 describes the nature of uncertainty and ambiguity and the evolution of beliefs of agents under ambiguity and presents Bayesian inference of the underlying TFP process that is used as the process generating ambiguity for agents. Section 3 describes a real business cycle model with ambiguity and ambiguity aversion while Section 4 presents the quantitative results. Section 5 is devoted to an analysis of the behavior of the risk-free rate and its estimates using industry-level TFP data while Section 6 concludes.

2 The sources of uncertainty

Uncertainty in this economy is assumed to be driven by the stochastic behavior of productivity growth. Specifically, there is a long-run average growth rate of productivity, $\bar{g}$, and a deviation from it, $x_{k,t+1}$, which is assumed to follow a persistent stochastic process. This specification of the technology process is similar to the models of long-run risk proposed by Bansal and Yaron (2004) and Croce (2014); see also Collard *et al.* (2018). However, the business cycle effect on productivity, $x_{k,t+1}$, is not observed directly. Specifically, the agent considers the model

$$g_{A,t+1} = \bar{g} + x_{t+1} + \sigma_A \epsilon_{A,t+1}, \tag{2.1}$$

$$x_{t+1} = \rho x_t + \sigma_x \epsilon_{x,t+1}, \tag{2.2}$$

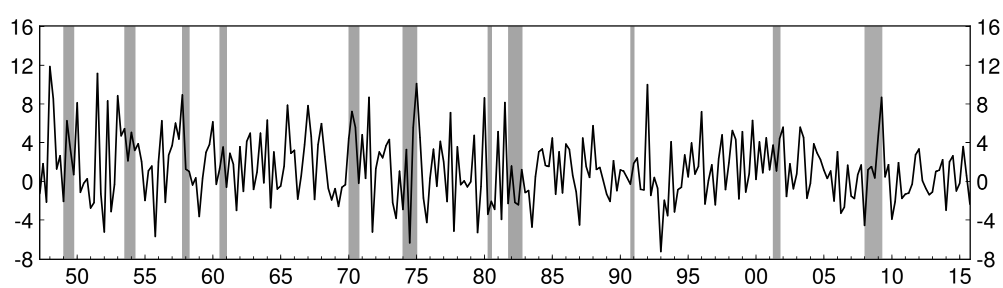
where $(\epsilon_{A,t+1}, \epsilon_{x,t+1})' \sim N(0, I_2)$ and I_2 is a 2×2 identity matrix. Here $\bar{g}$ denotes the long-run value for the technology growth, and the latent process x_{t+1} is the temporary deviation for this long-run value. Given these assumptions, next period's technology shock is written as

$$A_{t+1} = A_t \exp(g_{A_k,t+1}) = A_t \exp(\bar{g} + x_{k,t+1} + \sigma_{A_k} \epsilon_{A_k,t+1}). \tag{2.3}$$

According to this representation, the growth rate of the technology shock between t and $t+1$ evolves as a function of the permanent mean, $\bar{g}$, the temporary component $x_{k,t+1}$, and some noise. At time t, the agent has available observations on the current and past values of the technology shock, A_t. However, the agent does not know the process generating $x_{k,t}$ and forms beliefs about it, given prior beliefs at time 0 and the observations on the technology as A_s for $s = 1, \ldots, t$.

For the observations on (the growth rate of) the technology shock, A_t, we use seasonally adjusted data on total factor productivity (TFP) growth obtained from the Federal Reserve of San Francisco, see Fernald (2012) for details. The data on inputs, including capital, are used to produce a real-time, quarterly series on total factor productivity growth as the measured Solow residual. The advantages of these data are that they are adjusted to account for the changes in factor utilization and they are at the quarterly frequency unlike the typical annual TFP data. Figure 1 displays the growth rates of the factor utilization adjusted TFP series measured (at annualized rates) for the full sample together with NBER recession dates in shaded areas.

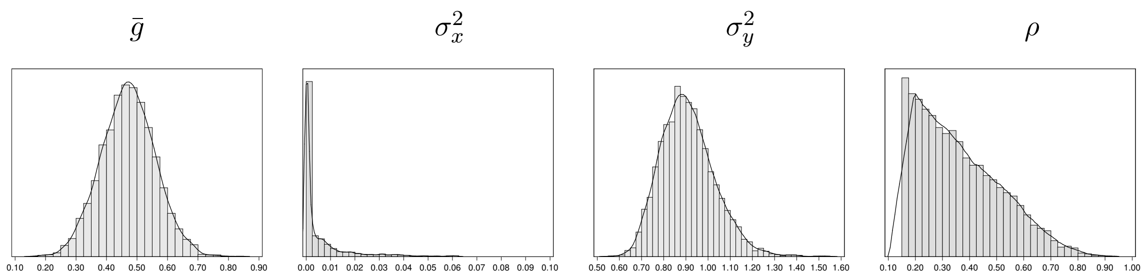
Figure 1: The growth rate of factor utilization adjusted TFP (in percentage terms) over the sample 1947:2-2015:4



Recently, many authors have observed that there has been a secular decline in TFP growth; see for example Gordon (2015), among others. This finding is also evident from Figure 1. Specifically, average TFP growth has declined in the post-1980's relative to the pre-1980's period from 1.78% to 0.88% for the adjusted TFP growth series, measured at annualized rates. There is also a decline in the variability of TFP growth after 1980 but this is not as large as the decline in the average quantities. Specifically, the standard deviation of adjusted TFP growth has fallen from 3.75% in the pre-1980's period to to 2.94% in the post-1980's period.[3]

As we seek to exploit the business cycle effects in the TFP growth process, we assume the existence of an agent who is situated in 1977, which implies that the agent has roughly 30 years of data to infer the parameters of the model in 2.1-2.2; see Collard *et al.* (2018) for a similar setup. This corresponds to the part of the data set prior to the occurrence of the potential (*ex-post*) structural change in the observed TFP process. We assume that the agent is a Bayesian learner regarding the distribution of the underlying parameters and infers the values of the persistence/volatility pairs using a Bayesian rule by combining prior beliefs with the data, i.e. prior distributions of the parameters together with the likelihood function to construct posterior distributions. We assume non-informative prior distributions reflecting the agent's ignorance about the model parameters. We display the posterior distributions of the model parameters in Figure 2.

Figure 2: Parameter distributions estimated using the TFP growth process



As shown in Figure 2 the distribution of the model parameters are precisely estimated except for the persistence parameter. Therefore, we assume that the agent can infer the parameters $\bar{g}, \sigma_x^2$ and σ_y^2 by using the posterior distributions. However, the ambiguity confronting by the agent stems from the persistence parameter (and therefore, from x_t) as the persistence parameter ρ covers a wide range of values between 0 and and 0.90 with high probability. This also indicates the difficulty to pinpoint the exact value of the persistence in the TFP growth process. To see this further, next, we estimate the models for given values of ρ covering this range, i.e. $\rho = 0.25, 0.30, 0.65, 0.70, 0.85, 0.90$.

[3]Similar findings hold for the unadjusted TFP growth series. Specifically, its growth rate has declined from 1.72% to 0.85% in the post-1980's relative to the pre-1980's while its standard deviation has declined from 4.07% to 2.78% across the two periods.

The results are displayed in Table 1.

Table 1: Posterior results for the model using TFP-util for different values of ρ using the sample of 1947-2 : 1977-4

ρ	0.25	0.30	0.65	0.70	0.85	0.90
$\bar{g}$	0.469 (0.086)	0.469 (0.086)	0.469 (0.086)	0.469 (0.086)	0.469 (0.086)	0.469 (0.086)
σ_A	0.945 (0.075)	0.946 (0.071)	0.949 (0.063)	0.950 (0.063)	0.952 (0.062)	0.953 (0.062)
σ_x	0.046 (0.120)	0.044 (0.110)	0.056 (0.080)	0.054 (0.073)	0.040 (0.049)	0.033 (0.040)
Max. Like.	-167.40	-167.40	-167.40	-167.40	-167.40	-167.40
Mar. Like.	-169.70	-169.73	-170.16	-170.20	-170.32	-170.38

Note: The results show the posterior means and posterior standard deviations (in parenthesis) of the model parameters in (2.1)-(2.2) (evaluated in percentage terms). The inference was carried out with 60,000 draws where the first 10,000 are used as burn-in sample. We kept every 5^{th} draw, which yields a sample of 10,000 draws from the ergodic distribution. Max. Like. refers to the maximized likelihood value of the models, whereas Mar. Like refers to the marginal likelihood value of the models, i.e. independent of the specific parameter values except the value of the ρ.

Table 1 provides important insights regarding the ambiguity the agent is facing. While the estimated value of the $\bar{g}$ remains unchanged, there are some rather minor changes in the estimates of the variance parameters. Regardless of the values of the ρ ranging from 0.20 to 0.90, the values of (logarithm of the) maximum likelihood, which is computed using the posterior mode of the distributions, do not change at all. Hence, different processes depending on the value of persistence parameter yield the same likelihood. We also compute the (logarithm of the) marginal likelihood values of the models. Note that the marginal likelihood is computed by integrating out the parameter distributions taking parameter uncertainty into account. Therefore, it provides a precise metrics of model probabilities as it is independent of the parameter values except the value of persistence parameter, ρ. The changes in the marginal likelihood values are very minor, only to the scale of 0.60 in terms of the log-differences. This implies that the agent faces ambiguity about the process stemming from the fact that the value of the persistence parameter cannot be inferred precisely.

2.1 Beliefs

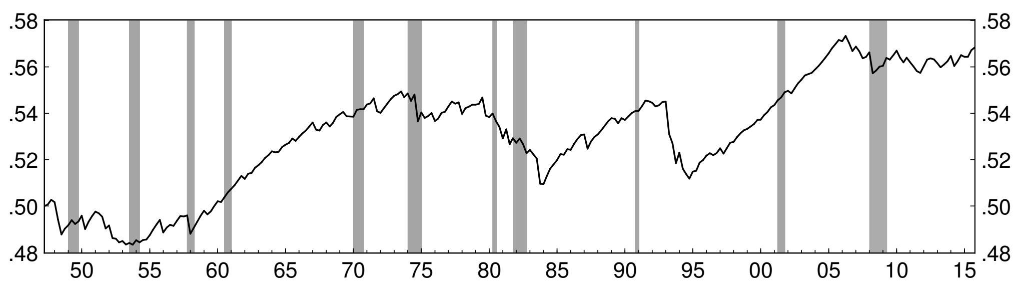
Following the evidence in the previous section, we model the ambiguity that agents are facing as follows. Agents are unsure about the value of the persistence parameter that determines the evolution of the latent productivity process. Specifically, they believe that the persistence parameter could be high (ρ_h) or low (ρ_l). Without loss of generality, we assume that the low persistence process is also a high variance process while the high persistence process has low variance. For a given $(x_{k,t}, \rho_k)$ and the current observations, the probability distribution over $g_{A_k,t+1}$ is given by

$$g_{A_k,t+1} \sim N(\bar{g} + \rho_k x_{k,t}, \sigma^2_{A_k} + \sigma^2_{x_k}). \tag{2.4}$$

which is the first-order uncertainty the agent confronts as discussed earlier.

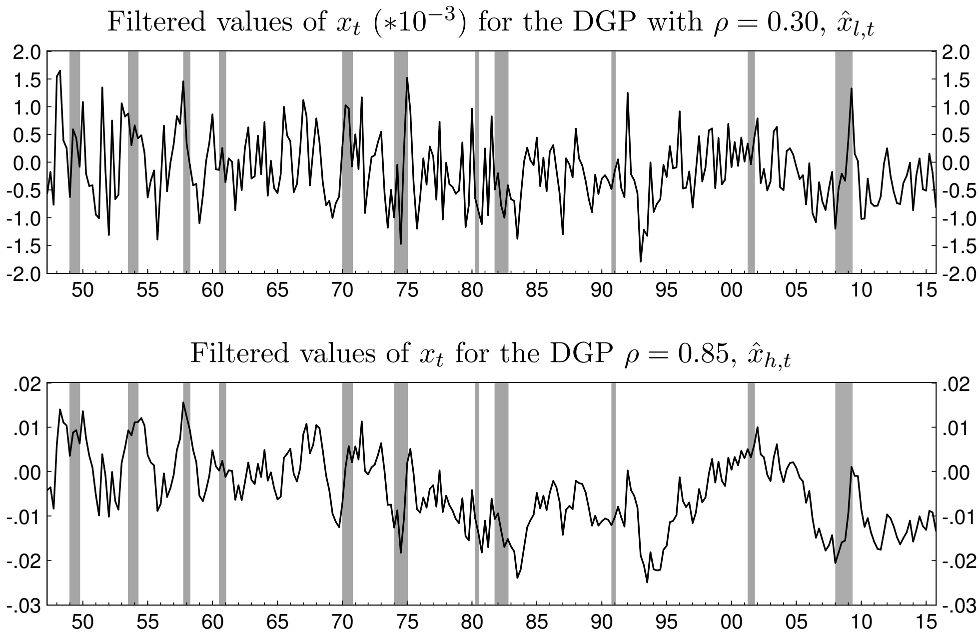
We now turn to a characterization of second-order uncertainty confronted by the agent, which arises from the fact that agent does not know which process $x_{k,t}$ is drawn from. First, given the parameter setup indicated in Table 1, we compute agent's belief about the true DGP denoted as η_t for the low persistence model with $\rho = 0.30$ relative to the one with $\rho = 0.85$. This can be computed using the agent's updating mechanism after observing data on the actual TFP growth process. We use the steady state Kalman filter to compute the beliefs, the details of which are provided in the next section. We compute the sequence of η_t's over the period starting from 1947:12 until 2015:4 using the parameter setup as shown in Table 1. Figure 3 displays the evolution of these computed beliefs.

Figure 3: Evolution of the probability of the model with $\rho = 0.30$ being the true DGP over the sample 1947:2:2015-4



In line with the results so far, the probabilities attached to each separate process vary in a band around 0.50-0.55, with some tendency to increase above this value towards the end of the sample. This suggests that there is very little learning that is occurring over the sample period, though we do see an increase in the probability attached to the low persistence process in the run-up to the 2008 global financial crisis. The two panels in Figure 4 further show the time series of the filtered means, $\hat{x}_{l,t}$ and $\hat{x}_{h,t}$ (in percentage terms) estimated using data on actual TFP growth.

Figure 4: Filtered values of the deviations, x_t, from the long-run over the sample 1947:2:2015-4



The filtered means tend to decline during the recessions of the 1970's and 1980's as well as during the global financial crisis of 2008. This decline is particularly severe for agents' beliefs regarding the cyclical mean of the high persistence process, $\hat{x}_{ht}$.

The support of the second-order distribution is a union of two component sets, $\{\rho_l x_{l,t} | x_{l,t} \in \Re\}$ and $\{\rho_h x_{h,t} | x_{h,t} \in \Re\}$. The agent's prior belief ascribes a measure to each component set, with the measure on the first component being given by $\eta_0 \times N(0, \sigma_0^2)$ and that on the second component by $(1 - \eta_0) \times N(0, \sigma_0^2)$. The agent updates her beliefs according to Bayes rule, based on the history of growth realizations and under the assumption that the economy conforms to one of the two processes described above. Let $\hat{x}_{k,t} \equiv E[x_{k,t} | g_{A_k,1}, \ldots, g_{A_k,t}]$ denote the expectation of $x_{k,t}$, conditional on the history of growth rates up to t, i.e. filtered values of x_t, if the beliefs were updated assuming $\rho = \rho_k$ is the data generating process. The agent's posterior beliefs then ascribes a measure on the first component set given by $\eta_t \times N(\hat{x}_{l,t}, \Omega_l)$ and that on the second by $(1 - \eta_t) \times N(\hat{x}_{h,t}, \Omega_h)$, where $\Omega_k, k = l, h$ denotes the steady state variance associated with the Kalman filter based on the process with $\rho = \rho_k$. Hence, the agent's posterior beliefs may be summarized by the tuple $\mu_t = (\hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t)$.

3 A real business cycle model with ambiguity aversion

We consider a one-sector economy where the production function of the firm is given by

$$y_t = k_t^a (A_t n_t)^{1-a}, \quad 0 < a < 1, \tag{3.1}$$

where A_t is the labor-augmenting technology shock as described in (2.3), k_t is the beginning-of-period capital stock and n_t is total hours. The firm's capital stock evolves as

$$k_{t+1} = (1 - \delta)k_t + i_t, \tag{3.2}$$

where i_t is gross investment and $0 < \delta < 1$ is the depreciation rate. Finally, we assume that investment is irreversible, $i_t \geq 0$. While it may be argued that investment irreversibility does not hold at the aggregate level, we choose to use this investment friction in place of the standard adjustment costs model, which is a reduced-form mechanism for generating frictions in the capital accumulation process. By contrast, the presence of investment irreversibility leads to an *endogenous* cost of adjustment that varies with uncertainty and information possessed by the firm; see Demers *et al.* (2003) for a further discussion.

As for the agents' preferences, following the findings in the previous section, we set out a dynamic, recursive version of the smooth ambiguity averse preferences as developed by Klibanoff *et al.* (2005, 2009), denoted as KMM hereafter. This model is based on the state space, which is the set of all observation paths emanating from an initial state s_0. Thus, the state at date t is denoted $s^t = (s_0, s_1, \cdots, s_t)$, where $s_t \in \Upsilon_t$. Agents choose consumption/investment plans f, each of which associates a payoff to the node s^t in the event tree. The agent is uncertain about the stochastic process governing the probabilities on the event tree. This uncertainty is indexed by the parameter $\theta \in \Theta$, which denotes the set of unobservable parameters. The probability that the next observation will be s_{t+1}, given that the node s^t has been reached on the event tree, is given by $\pi_\theta(s_{t+1}|s^t)$. The agent further has a prior $\mu(\theta)$ for $\theta \in \Theta$. Using the representation in KMM, the recursive smooth ambiguity preferences over plans f at the node s^t are updated and represented as

$$V_{s^t}(f) = u(f(s^t)) + \beta\phi^{-1}\left[\int_\Theta \phi\left(\int_{\Upsilon_{t+1}} V_{(s^t,s_{t+1})}(f)d\pi_\theta(s_{t+1}|s^t)\right)d\mu(\theta|s^t)\right], \tag{3.3}$$

where $V_{s^t}(f)$ is a recursively defined direct value function, $u(\cdot)$ characterizes attitudes towards risk, β is a discount factor, $\phi(\cdot)$ is a function characterizing the agent's ambiguity attitude, and $\mu(\cdot|s^t)$ denotes the Bayesian posterior. A concave function, ϕ, characterizes ambiguity aversion, which is defined to be an aversion to mean-preserving spreads in the distributions of expected utility values. The model also does not, in general, allow a reduction between the second-order beliefs μ and the first-order probabilities denotes by π_θ in terms of the predictive distribution for s_{t+1}, given θ; such a reduction occurs only in the case of a linear ϕ which represents an ambiguity neutral Bayesian expected utility maximizer.

3.1 The social planner's problem

We consider the model as the social planner's problem for this economy. Given the stochastic growth in the technology shock, the state variables for the social planner's problem consist of the initial capital k_t, initial beliefs μ_t and the level of the technology shock, A_t. Thus, the generic social planner's problem is given by

$$J(k_t,\mu_t,A_t) = \max_{c_t,n_t,i_t}\left\{u(c_t,l_t) + \beta\phi^{-1}\left[E_{\mu_t}\phi\left(E_{x_t}J(k_{t+1},\mu_{t+1},A_{t+1})\right)\right]\right\} \tag{3.4}$$

subject to $c_t + i_t \leq k_t^a(A_t n_t)^{1-a},\ k_{t+1} = (1-\delta)k_t + i_t,\ l_t + n_t \leq 1,\ i_t \geq 0$ and the law of motion for beliefs which we discuss in the next section.

Since the technology shock A_t is nonstationary, we will consider the transformed value function in terms of stationary variables. Allowing for an endogenous hours choice, a stationarity inducing transformation exists under two different sets of assumptions for the utility function and the smooth ambiguity aversion functions, namely, a power-power specification for risk aversion and ambiguity aversion and the log-exponential specification.[4] Another possibility is to use the homothetic representation of smooth ambiguity preferences employed by Ju and Miao (2012) that allows for a three-way separation among risk aversion, ambiguity aversion, and intertemporal substitution. However, our aim is to examine the properties of the smooth ambiguity model proposed by Klibanoff *et al.* (2005, 2009).[5]

Define

$$\left\{\hat{c}_t, \hat{i}_t, \hat{k}_t, \hat{y}_t\right\} = \left\{\frac{c_t}{A_t}, \frac{i_t}{A_t}, \frac{k_t}{A_t}, \frac{y_t}{A_t}\right\}$$

and assume that $\hat{J}(\hat{k}, \hat{\mu}_t)$ satisfies $J(k_t, \mu_t, A_t) = \hat{J}(\hat{k}, \hat{\mu}_t)A_t^{(1-\gamma)\nu}$. The state variables for the social planner's problem are given by the existing stock of capital, $\hat{k}_t$, and initial beliefs, $\hat{\mu}_t$. Substituting for the power-power specification into the representation for $\hat{J}$, the *indirect value function* is given by

$$\hat{J}(\hat{k}_t, \hat{\mu}_t) = \max_{\hat{c}_t, n_t, \hat{i}_t} \left\{ \frac{(\hat{c}_t^{\nu} l_t^{1-\nu})^{1-\gamma}}{1-\gamma} + \frac{\beta}{1-\gamma}\left[E_{\hat{\mu}_t}\left(E_{x_t}\hat{J}(\hat{k}_{t+1}, \hat{\mu}_{t+1})\exp(\nu(1-\gamma)g_{A,t+1})\right)^{1-\alpha}\right]^{\frac{1}{1-\alpha}}\right\}$$

subject to $\hat{c}_t + \hat{i}_t \leq \hat{k}_t^a n_t^{1-a}$, $\exp(g_{A,t+1})\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \hat{i}_t$, $l_t + n_t \leq 1$, $\hat{i}_t \geq 0$ and the law of motion for beliefs to be discussed below. In this specification, we restrict $0 < \gamma < 1$ and $\alpha > \gamma$, which ensures ambiguity aversion.[6],[7]

---

[4] Specifically, the power-power representation is given by (i) $u(c, l) = (c^{\nu} l^{1-\nu})^{1-\gamma}/(1-\gamma), \gamma \geq 0$, $0 \leq \nu \leq 1$ and $\phi(x) = x^{1-\alpha}/(1-\alpha), \alpha \geq 0$, where using $y = \phi(x)$, $\phi^{-1}(y) = [(1-\alpha)y]^{\frac{1}{1-\alpha}}$ while the log-exponential specification is defined as (ii) $u(c, l) = \ln(c) + \ln(l)$ and $\phi(x) = -\exp(-\alpha x)/\alpha$. See Section A of the Online Appendix.

[5] In the Ju and Miao (2012) framework, the problem is merely a portfolio choice problem in a simple version of the Lucas asset pricing model where the nonstationarity arises from growth in the exogenous dividend process, and there is no endogenous capital accumulation. In their portfolio choice problem, Ju and Miao (2007) are able to "guess" a specific functional form for the indirect value function and the policy function for consumption in terms of the wealth process for the consumer. In our case, such explicit "guesses" for the indirect value function entail additional assumptions on the production function, namely, constant returns to scale, and 100% depreciation along with no other frictions such as investment irreversibility or adjustment costs in investment.

[6] The case with $\gamma > 1$ is not defined for the power-power specification, as the indirect value function becomes negative in this case. However, as the parameter γ corresponds to the elasticity of intertemporal substitution for deterministic consumption paths, this parameter is typically assumed to be less than one in most business cycle studies, as we discuss in the next section.

[7] It is possible to replicate the solution to the social planner's problem in a recursive competitive equilibrium where the representative consumer makes consumption and labor supply choices and holds shares and bonds in the firm while all production and capital accumulation decisions are made by value-maximizing firms. In this setup, the presence of an irreversibility constraint is equivalent to assuming that there are no resale markets for used capital; see Altug and Labadie (2008), Ch. 10 for a discussion. Also see Kaltenbrunner and Lochstoer (2010) for a similar

Recall that the initial beliefs of the agent are specified by the triple $\mu_t = (\hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t)'$. Denote by $\hat{x}^{(i)}_{k,t+1}, i = l, h, k = l, h$, the agent's forecast for the one-period ahead update using a Kalman filter which takes the model with $\rho = \rho_i$ as the data generating process, when the data is actually generated by the $\rho = \rho_k$ process. Correspondingly, $\eta^{(l)}_{t+1}$ (respectively, $\eta^{(h)}_{t+1}$) is the posterior probability that the low persistence process is the correct model when the low (high) persistence process is the correct model. Under the assumption that consumption, investment and the capital stock normalized by the level of the technology shock, $\hat{c}_t = c_t/A_t, \hat{i}_t = i_t/A_t$, and $\hat{k}_t = k_t/A_t$, are stationary random variables, the indirect value function can now be defined as:

$$
\begin{aligned}
\hat{J}(\hat{k}_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) = \max_{\hat{c}_t, n_t, \hat{i}_t} \Bigg\{ & \frac{(\hat{c}_t^{\nu} l_t^{1-\nu})^{1-\gamma}}{1-\gamma} + \\
& \beta \Bigg[\eta_t E_{\hat{x}_{l,t}} \left(E_{x_{l,t}} \hat{J}(\hat{k}^{(l)}_{t+1}, \hat{x}^{(l)}_{h,t+1}, \hat{x}^{(l)}_{l,t+1}, \eta^{(l)}_{t+1}) \exp((1-\gamma) g_{A_l,t+1}) \right)^{1-\alpha} + \\
& (1-\eta_t) E_{\hat{x}_{h,t}} \left(E_{x_{h,t}} \hat{J}(\hat{k}^{(h)}_{t+1}, \hat{x}^{(h)}_{h,t+1}, \hat{x}^{(h)}_{l,t+1}, \eta^{(h)}_{t+1}) \exp((1-\gamma) g_{A_h,t+1}) \right)^{1-\alpha} \Bigg]^{\frac{1}{1-\alpha}} \Bigg\}
\end{aligned}
\tag{3.5}
$$

subject to $\hat{c}_t + \hat{i}_t \leq \hat{k}_t^a n_t^{1-a}$, $\exp(g_{A,t+1})\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \hat{i}_t$, $l_t + n_t \leq 1$, and $\hat{i}_t \geq 0$ and given the laws of motion for beliefs $x^{(k)}_{j,t+1}$, $j = l, h$ together with the condition determining the evolution of $\eta^{(k)}_{t+1}$ for $k = l, h$ described in the Appendix, Section A.

decentralization scheme in the context of an economy with non-expected utility preferences.

3.1.1 The optimality conditions

Define Υ_t and $\xi_{k,t}$ as

$$\begin{aligned}
\Upsilon_t \quad &= \quad \left[\sum_{k=l,h} \eta_{k,t} E_{\hat{x}_{k,t}} \left(E_{x_{k,t}} \hat{J}(\hat{k}^{(k)}_{t+1}, \hat{x}^{(k)}_{h,t+1}, \hat{x}^{(k)}_{l,t+1}, \eta^{(k)}_{t+1}) \exp(\nu(1-\gamma) g_{A,t+1}) \right)^{-\alpha} \right] \div \\
&\quad \left[\sum_{k=l,h} \eta_{k,t} E_{\hat{x}_{k,t}} \left(E_{x_{k,t}} \hat{J}(\hat{x}^{(k)}_{h,t+1}, \hat{x}^{(k)}_{l,t+1}, \eta^{(k)}_{t+1}) \exp(\nu(1-\gamma) g_{A,t+1}) \right)^{1-\alpha} \right]^{\frac{-\alpha}{1-\alpha}} \qquad (3.6) \\
\xi_{k,t} \quad &= \quad \left[\left(E_{x_{k,t}} \hat{J}(\hat{k}^{(k)}_{t+1}, \hat{x}^{(k)}_{h,t+1}, \hat{x}^{(k)}_{l,t+1}, \eta^{(k)}_{t+1}) \exp(\nu(1-\gamma) g_{A,t+1}) \right)^{-\alpha} \right] \div \\
&\quad \left[\sum_{k=l,h} \eta_{k,t} E_{\hat{x}_{k,t}} \left(E_{x_{k,t}} \hat{J}(\hat{x}^{(k)}_{h,t+1}, \hat{x}^{(k)}_{l,t+1}, \eta^{(k)}_{t+1}) \exp(\nu(1-\gamma) g_{A,t+1}) \right)^{1-\alpha} \right]^{\frac{-\alpha}{1-\alpha}}. \qquad (3.7)
\end{aligned}$$

Let λ_t denote the Lagrange multiplier on the aggregate resource constraint and φ_t the multiplier on the non-negativity constraint. Using the expressions for Υ_t and $\xi_{k,t}$, the first-order conditions with respect to $\hat{c}_t, l_t, \hat{i}_t$ are given by

$$\nu(\hat{c}_t^{\nu} l_t^{1-\nu})^{-\gamma} (\hat{c}_t / l_t)^{\nu-1} = \lambda_t, \qquad (3.8)$$

$$(1-\nu)(\hat{c}_t^{\nu} l_t^{1-\nu})^{-\gamma} (\hat{c}_t / l_t)^{\nu} = (1-a)\lambda_t \hat{k}_t^a n_t^{-a}, \qquad (3.9)$$

$$\lambda_t - \varphi_t = \Upsilon_t \sum_{k=l,h} \eta_{k,t} \left[\xi_{k,t} E_{x_{k,t}} \left(\hat{J}_1(\hat{k}^{(k)}_{t+1}, \hat{x}^{(k)}_{h,t+1}, \hat{x}^{(k)}_{l,t+1}, \eta^{(k)}_{t+1}) \exp((\nu(1-\gamma)-1) g_{A,t+1}) \right) \right]. \qquad (3.10)$$

Finally, the envelope condition is given by

$$\hat{J}_1(\hat{k}_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) = \lambda_t a \hat{k}_t^{a-1} n_t^{1-a} + (1-\delta)(\lambda_t - \varphi_t). \qquad (3.11)$$

The conditions (3.8-3.9) simplify to yield the condition describing the intratemporal substitution in consumption and labor supply as

$$\frac{1-\nu}{\nu} \frac{\hat{c}_t}{l_t} = (1-a) \hat{k}_t^a n_t^{-a}. \qquad (3.12)$$

Given a solution for $\hat{c}_t$ as a function of $(\hat{k}_t, \hat{\mu}_t)$, this condition can be solved for current l_t for each $\hat{k}_t$ and beliefs $\hat{\mu}_t$.

Define Υ^i_{t+1} and $\xi^i_{k,t+1}$, $i = 0, 1$ by the relevant expressions evaluated at the values of $\hat{i}_{t+1} = 0$ and $\hat{i}_{t+1} > 0$, respectively. Using conditions (3.10) and (3.11), consider a version of the envelope condition that holds at time $t + 1$ as

$$\hat{J}_1(\hat{k}_{t+1}, \hat{x}_{h,t+1}, \hat{x}_{l,t+1}, \eta_{t+1}) = \lambda_{t+1} a \hat{k}^{a-1}_{t+1} n^{1-a}_{t+1} + (1-\delta) \min \Bigg(\lambda_{t+1}, \Upsilon^0_{t+1} \sum_{k=l,h} \eta_{k,t+1} \left[\xi^0_{k,t+1} E_{x_{k,t+1}} \left(\frac{\hat{J}_1((1-\delta)\hat{k}_{t+1}, \hat{x}^{(k)}_{h,t+2}, \hat{x}^{(k)}_{l,t+2}, \eta^{(k)}_{t+2}) \exp((\nu(1-\gamma)-1) g_{A_k,t+2})}{\lambda_{t+1}} \right) \right] \Bigg) \tag{3.13}$$

The envelope condition provides an expression for the marginal value of capital next period. When there is an irreversibility constraint, the marginal value of capital accounts for the fact that the irreversibility constraint may be binding next period. This occurs when λ_{t+1} is greater than the expected value of the marginal value of capital next period, where the expectation reflects the existence of the distortions implied by the presence of ambiguity and ambiguity aversion. It is this aspect of the framework with irreversible investment that leads to an *endogenous risk premium* or an option value to wait and creates a non-linear mechanism for the transmission of technology shocks in a simple Real Business Cycle framework.

To further understand the distortions inherent in the model with ambiguity and ambiguity aversion, consider the conditions describing the optimal choice of investment at time t as

$$\lambda_t = \Upsilon^1_t \sum_{k=l,h} \eta_{k,t} \left[\xi^1_{k,t} E_{x_{k,t}} \left(\hat{J}_1(\hat{k}^{(k)}_{t+1}, \hat{x}^{(k)}_{h,t+1}, \hat{x}^{(k)}_{l,t+1}, \eta^{(k)}_{t+1}) \exp((\nu(1-\gamma)-1) g_{A_k,t+1}) \right) \right] \quad \text{if } \hat{i}_t > 0, \tag{3.14}$$

$$\lambda_t > \Upsilon^0_t \sum_{k=l,h} \eta_{k,t} \left[\xi^0_{k,t} E_{x_{k,t}} \left(\hat{J}_1((1-\delta)\hat{k}_t, \hat{x}^{(k)}_{h,t+1}, \hat{x}^{(k)}_{l,t+1}, \eta^{(k)}_{t+1}) \exp((\nu(1-\gamma)-1) g_{A_k,t+1}) \right) \right] \quad \text{if } \hat{i}_t = 0, \tag{3.15}$$

where we substitute for the envelope condition given by (3.13).

In our framework, ambiguity arises from the nature of the processes that generate TFP growth which, combined with the ambiguity averse preferences, leads to an endogenous tilting or distortion of the posterior distributions that signify agent's subjective beliefs about the validity of a given process describing the external environment. The optimality conditions in (3.14-3.15) depend on the distortion factors Υ_t and $\xi_{k,t}$. While Υ_t does not affect the second-order distributions appearing in the optimality conditions (3.14-3.15), the distortion term defined by $\xi_{k,t}$ does. In this expression, $\xi_{k,t}$ depends on expectations that are taken with respect to the distribution of $x_{k,t}$, conditional on information of the history the technology shock $(g_{A,t}, g_{A,t-1}, \cdots, g_{A,0})$ and, hence, is random from the view of the agent's subjective beliefs at date t. The function $\xi_{k,t}$ may be viewed as the factor that create the endogenous tilting or distortion in agents' beliefs due to ambiguity aversion. In the case of ambiguity aversion with $\alpha > \gamma > 0$, the distortion puts greater weight (relative to a pure Bayesian decision-maker) on the probability distributions of the $x_{k,t}$ that are associated with lower expected continuation values, $E_{x_{k,t}} \hat{J}(\hat{k}^{(k)}_{t+1}, \hat{x}^{(k)}_{h,t+1}, \hat{x}^{(k)}_{l,t+1}, \eta^{(k)}_{t+1})$ for $k = l, h$. Thus, we may view the impact of the $\xi_{k,t}$ as shaping the "as if" beliefs of the agent, that is, the (probabilistic) belief that supports the action chosen in equilibrium. In the absence of ambiguity aversion, the agent behaves as a pure Bayesian decision-maker who is uncertain about the temporary component of TFP growth x_t, and has beliefs that are just a mixture of the probability distributions for $x_{kt}, k = l, h$, as discussed above.

4 Quantitative results

We present the unconditional business cycle moments for the full sample between 1947:12015:4 and the restricted sample between 1978:1-2015:4 in Table 2. All series are Hodrick-Prescott filtered versions of the original series. The data on output, consumption, investment, and hours worked are obtained from the Federal Reserve Bank of St. Louis database (FRED). The output, y, consumption, c, and investment, i series are seasonally adjusted and measured in chained 2009 dollars. Investment refers to total private investment and the hours worked series is an index of total hours worked in the nonfarm business sector, seasonally adjusted with the 2009 value equal to 100. Table 2 shows that the volatility of the endogenous variables are considerably lower in the restricted sample while the correlations between the different variables are higher.[8] We also observe that the correlations of labor productivity $p = y/h$ with output, consumption, and investment are very small for the full sample but these become negative for the restricted sample.

Table 2: Unconditional business cycle moments

	Full Sample - 1947:1-2015:4					Restricted Sample - 1980:1-2015:4				
	Investment-Output Ratio, i/y: 0.143					Investment-Output Ratio, i/y: 0.157				
	Standard deviations									
	y	c	i	h	p	y	c	i	h	p
	1.624	1.262	7.442	1.925	0.934	1.294	1.076	6.284	1.783	0.930
	Correlations									
	y	c	i	h	p	y	c	i	h	p
y	1.000	0.774	0.841	0.886	-0.064	1.000	0.866	0.913	0.865	-0.266
c		1.0000	0.694	0.658	-0.068		1.000	0.730	0.768	-0.267
i			1.0000	0.762	-0.100			1.000	0.817	-0.296
h				1.0000	-0.5393				1.000	-0.715
p					1.000					1.000

Note: This table shows the average investment-output ratio and the unconditional second moments of the HP-filtered series on output, y, consumption, c, investment, i, hours worked, h, and labor productivity, p.

4.1 Simulation results

In this section, we compute the unconditional moments for all of the series by randomly drawing sample of shocks from the high and low persistence processes to generate a pseudo observation on the growth rate of technology $g_{A,t+1}$ at each date. We continue to assume that the agent does not know which process the realization of the technological growth shock is coming from and must make inferences from observations on the growth rate of technology about the nature of the process from which such observations are drawn. Given initial conditions $\hat{k}_0 = k_s, \hat{x}_{l,0} = 0, \hat{x}_{h,0} = 0, \eta_0 = 0.5$, we use the laws of motion for the capital together with the linear Gaussian filter to determine the evolution of the capital stock and beliefs along the agent's optimal path. These will constitute

[8] This finding might be taken as evidence of the "Great Moderation" that various authors such as Stock and Watson (2002) have documented for the post-1980's era.

Table 3: Parameter Values

β	Subjective discount factor	0.988
γ	Coefficient of risk aversion	0.5
α	Coefficient of ambiguity aversion	0.8, 5
a	Capital share	0.3
δ	Depreciation rate	0.025

Note: This table reports parameter choices used in the simulations of the models. The calibration is at the quarterly frequency. The share of leisure is implicitly computed from the intratemporal marginal rate of substitution between consumption and leisure, assuming a steady value of hours worked $n_{ss} = 1/3$.

the endogenous state variables for the model. Notice that updating the capital stock depends on using the optimal policy functions for investment and hours worked as $\hat{i}_t = g(\hat{k}_t, \hat{x}_{lt}, \hat{x}_{ht}, \eta_t)$ and $h_t = h(\hat{k}_t, \hat{x}_{lt}, \hat{x}_{ht}, \eta_t)$ evaluated at the current state $(\hat{k}_t, \hat{x}_{lt}, \hat{x}_{ht}, \eta_t)$. Given the simulated values of $\hat{i}_t, h_t$ and $g_{A,t+1}$, output and consumption today together with next period's capital stock are obtained from the production function, the resource constraint and the law of motion for capital. To generate next period's beliefs which will form next period's state vector, we update current beliefs $\mu_t = (\hat{x}_{lt}, \hat{x}_{ht}, \eta_t)$ given the new observation $g_{A,t+1}$ using the Kalman filter.

The parameter values used in the simulations are standard to the business cycle literature. Specifically, the capital share is set at 0.36 and the depreciation rate at the quarterly frequency at 0.025. The share of leisure in preferences denoted by the parameter ν is given by 0.3663. This is based on the steady state values of the model where the share of working time is set at 1/3, consistent with the finding that households spend one-third of their time working. The discount rate is set at $\beta = 0.988$, which is slightly lower than the value assumed by Prescott (1986) and implies an annual interest rate of around 4.8%. The estimates of Table 1 imply that the average growth rate of the technology shock at the quarterly frequency is 0.47%, which is slightly higher but consistent with the average growth rate of technology based on the Markov switching model reported in Jahan-Parvar and Liu (2014) as well as earlier estimates in Kaltenbrunner and Lochstoer (2010) and Croce (2014).

Table 4 displays the results obtained by simulating the model with ambiguity and ambiguity aversion under alternative technology shock processes. The decision rules for the transformed problem are used to generate levels for the nonstationary series, which are then detrended using a Hodrick-Prescott filter for quarterly series. The simulated moments are based on 1,000 simulations of 400 periods, with a burn-in sample of 200 periods. In these simulations, the estimates of the technology shock processes are obtained by using uninformative priors as reported in Table 1. While the simulations are conducted separately for the low persistence/high variance and high persistence/low variance processes by drawing realizations of the shocks from these processes, the agent is assumed not to know from which process the underlying technology shocks is coming from.

In the smooth model of ambiguity, ambiguity aversion is inferred from properties of the functions $\phi(\cdot)$, and is measured by the parameter α for the specification of preferences used in this paper.[9] Risk aversion, as usual, is inferred from the properties of the $u(\cdot)$, and is measured by the parameter γ.[10] In other recent analyses which allow for the separation of risk aversion and intertemporal substitution, the elasticity of intertemporal substitution (EIS) is typically set to be greater than one. For evidence on this point, see Vissing-Jørgensen and Attanasio (2003), though the evidence is mixed. An EIS greater than one corresponds to a value of γ less than one in our specification. Ju and Miao (2012) set the ambiguity aversion parameter based on thought experiments in a specification that allows for a three-way separation between risk aversion, intertemporal substitution and ambiguity aversion. In other analyses, the ambiguity parameter is chosen to match various endogenous moments in the data, e.g. Collard *et al.* (2018) choose the ambiguity aversion parameter to match the risk-free rate of 1.5%, averaged over the period 1978-2011, while Liu and Zhang (2018) set it to match the mean equity premium in the data. In our quantitative analysis, we employ the parameter configurations of $\gamma = 0.5, \alpha = 0.8$, $\gamma = 0.5, \alpha = 5$ and $\gamma = 0.8, \alpha = 5$, which allows for an EIS greater than unity together with variation across the values of γ and α.

Table 4 reports the results for the main model with ambiguity and learning. We find that the model is, in general, able to replicate standard business cycle facts. Here (i) the volatility of investment is nearly three times that of output, (ii) output growth is more variable than consumption,

[9] More generally, Klibanoff *et al.* (2005) show that ambiguity aversion is defined as an aversion to mean preserving spreads in the distribution of expected utilities induced by agent's prior beliefs under a specific action, which corresponds to $-\phi''(x)x/\phi'(x)$.

[10] The specification of preferences employed here does not allow for a separation of risk aversion and the elasticity of intertemporal substitution (IES).

and (iii) the magnitudes of the simulated standard deviations match the standard deviations of the actual series for the post-1980's sample period. However, the model does much more poorly in matching the variability of hours, which is a standard result in the business cycle literature; see Hansen (1985) or Altug (1989). The correlations of the simulated series capture the pro-cyclicality of consumption and investment, though lower values of γ lead to lower correlations between the simulated consumption series, on the one hand, and simulated output, investment and hours, on the other, relative to the data.

In the second panel of Table 4, we increase the value of α to 5, holding the value of γ constant at 0.5. In this case, under the assumption that the TFP process is drawn from the low persistence/high variability process, the variability of series such as investment, hours, and output tend to increase slightly relative to the case with $\gamma = 0.5, \alpha = 0.8$ but they tend to fall if the TFP process is drawn from the high persistence/low variability process. Thus, increasing the coefficient of ambiguity aversion, while holding the value of γ constant, tends to reduce the difference in the response of the endogenous series depending on whether the TFP process is drawn from the high persistence/low volatility versus the low persistence/high volatility process, without changing the overall magnitudes significantly compared to the case with $\gamma = 0.5, \alpha = 0.8$. Thus, under greater ambiguity aversion, agents engage in precautionary saving behavior by smoothing their responses under the high persistence/low volatility process relative to the low persistence/high volatility process. The precautionary behavior of agents under ambiguity aversion is also noted by Cagetti *et al.* (2002), Jahan-Parvar and Liu (2014) and Liu and Zhang (2018), amongst others. On the other hand, increasing the value of the ambiguity aversion parameter has a minor impact on the business cycle moments overall. A similar finding is reported by Liu and Zhang (2018).[11]

In an earlier analysis, Tallarini (2000) and Backus *et al.* (2015) studied the role of uncertainty/ambiguity aversion in real business cycle models of the type studied here. First, although an increase in α is associated with an increase in ambiguity aversion, their analysis suggests that this factor tends to have a minor impact on the behavior of quantities if agents can optimally respond to such uncertainty through their choice of investment and hours to smooth their consumption over

[11] In their sensitivity analysis, they show that changing the value of their ambiguity aversion parameter, η, has only a minor effect on the volatilities of consumption and investment. See their Table 7.

Table 4: Simulation Results: Ambiguity about the Persistence/Variability of the Unobserved Component of TFP Growth

	$\gamma=0.5,\alpha=0.8$					$\gamma=0.5,\alpha=5$					$\gamma=0.8,\alpha=5$				
Simulations conditional on $\rho_k=0.85$															
Standard deviations															
	y	c	i	h	p	y	c	i	h	p	y	c	i	h	p
	1.363	0.603	6.403	0.715	0.777	1.354	0.598	6.370	0.711	0.772	1.242	0.602	5.127	0.532	0.781
Correlations															
	y	c	i	h	p	y	c	i	h	p	y	c	i	h	p
y	1.000	0.662	0.939	0.905	0.921	1.000	0.662	0.939	0.906	0.921	1.000	0.858	0.952	0.920	0.964
c		1.000	0.368	0.282	0.902		1.000	0.368	0.282	0.902		1.000	0.661	0.587	0.964
i			1.000	0.994	0.732			1.000	0.994	0.732			1.000	0.994	0.837
h				1.000	0.668				1.000	0.669				1.000	0.781
p					1.000					1.000					1.000
i/y					0.172					0.173					0.166
Simulations conditional on $\rho_k=0.30$															
Standard deviations															
	y	c	i	h	p	y	c	i	h	p	y	c	i	h	p
	1.341	0.588	6.318	0.706	0.761	1.343	0.590	6.327	0.706	0.763	1.221	0.589	5.057	0.524	0.766
Correlations															
	y	c	i	h	p	y	c	i	h	p	y	c	i	h	p
y	1.000	0.660	0.940	0.907	0.921	1.000	0.661	0.940	0.907	0.921	1.000	0.858	0.953	0.921	0.964
c		1.000	0.368	0.284	0.900		1.000	0.369	0.284	0.901		1.000	0.664	0.590	0.964
i			1.000	0.994	0.735			1.000	0.994	0.734			1.000	0.994	0.839
h				1.000	0.672				1.000	0.672				1.000	0.785
p					1.000					1.000					1.000
i/y					0.172					0.174					0.166

Note: The model is simulated based on the decision rules for the main model with ambiguity where the agent cannot distinguish perfectly between two processes with persistence $\rho_l = 0.30$ and $\rho_h = 0.85$. The parameters characterizing the shock processes are derived from the estimates in Table 1 and assume that $\bar{g}_h = 0.00469$ and $\rho_h = 0.85$, $\sigma_{A_h} = 0.00952$, $\sigma_{x_h} = 0.0004$ for the high persistence process and $\rho_l = 0.30$, $\sigma_{A_l} = 0.00946$, $\sigma_{x_l} = 0.00044$ for the low persistence process.

time. In the Online Appendix, Section B, we generate a log-linear approximation to the model under the assumption of known persistence ($\eta_t = 0$) for the TFP growth process. There we show that the dynamics of the capital stock is unaffected by the ambiguity aversion parameter, α, as are the coefficients on the filtered mean of the temporary component of TFP growth, $\hat{x}_t$.[12] These results do not carry over directly to the more general version of the model, as ambiguity continues to play a role in the long-run in that case. Nevertheless, we would expect the result obtained under the assumption of known persistence ($\eta_t = 0$) to provide insight into the role of ambiguity aversion even in the case with $\eta_t > 0$.[13]

Finally, in the third panel of Table 4, we increase the value of γ to 0.8, holding the value of α at 5. We observe that the standard deviations of output, investment and hours fall, and the correlations increase. Since the ambiguity aversion parameter is constant, we may attribute this finding to an increase in risk aversion and a decline in intertemporal substitution. Strzalecki (2013) shows that there is interdependence between ambiguity and the timing of the resolution of uncertainty in models of ambiguity aversion, and that a quantitative assessment is required to disentangle the importance of two effects, which may depend on the calibrated parameters in applications such as ours. In our case, to the extent that an increase in γ reflects the decline in the willingness of consumers to substitute consumption across periods, we observe a commensurate decline in the volatility of all of the key macroeconomic aggregates as well as an increase in their co-movement.

The behavior of the investment-output ratio provides another gauge of the impact of ambiguity aversion. This is close to the value reported for the data, and suggests that the model is able to reconcile the average value of 0.157 for the post-1980's. The investment-output ratio is higher for simulations based on draws of the low persistence/high volatility process: since good realizations of the technology shocks are not expected to persist, agents form a buffer by increasing their investment relative to output to smooth their consumption over time. Similarly, the investment-output ratio

[12] Our approach follows Backus *et al.* (2015), who perform a log-linear approximation for the solution of the solution planner's problem in the non-expected utility and smooth ambiguity cases without learning.

[13] As Tallarini (2000) and Backus *et al.* (2015) have argued, this result reflects the full insurance/complete markets assumption that underlies the social planner's problem used to generate the business cycle observations. To demonstrate this result, it is possible to formulate a recursive complete contingent claims equilibrium which supports the allocations in the social planner's problem. In this equilibrium, consumers can insure against the future state as well as the distribution from which that state is drawn.

tends to increase for the simulations reported in the second panel of Table 4, as greater ambiguity aversion leads agents to engage in greater precautionary savings behavior. Finally, the third panel of Table 4 shows that the investment-output ratio falls with an increase in γ, suggesting that lower volatility of the key macroeconomic aggregates goes hand in hand with a lower investment-output ratio.

4.2 Informative priors

In this section, we provide simulations using the estimates of the underlying TFP processes based on more informative priors. Notice that the beliefs, η_t, that the agent form on the specific data generating process are affected by the parameter values. As the agent is a Bayesian learner, informative prior beliefs/distributions on model parameters could assist to resolve the (second-order) uncertainty on the specific data generating process. Specifically, priors increasing the signal-to-noise ratio as computed by the ratio of the variances of transitory and permanent components, σ_x^2/σ_g^2, could improve the inference on the transitory component. This, in turn, would increase the ability of the agent to discriminate between the low and high persistent processes with the increasing informativeness of the transitory process. Therefore, this may be seen as a shock to the agent's "confidence" regarding the nature of the underlying TFP process as the informative content of the transitory component increases with the increasing signal-to-noise ratio.[14]

We display the details of the prior settings and the resulting evolution of beliefs, η_t in the Appendix, Section B. We report the results for the case with $\gamma = 0.5, \alpha = 0.8$.[15] We consider the estimates of the shock processes displayed in Panels B and E of Table B.1 as examples of estimates based on progressively more informative priors. The results in Table 5 compared to those in the baseline case reported in Table 4 show that when agents are *more* confident about the process governing the cyclical component of TFP, as reflected by the evolution of the beliefs η

[14] Jewitt and Mukerji (2017) provide a general definitions of the notion of ordering ambiguous acts. According to one definition, a more ambiguous act makes an ambiguity averse decision maker (DM) worse off but does not affect the welfare of an ambiguity neutral DM. Second, a more ambiguous act adversely affects a more ambiguity averse DM more, as measured by the compensation required to switch acts. In a portfolio choice problem with model uncertainty regarding the conditional mean of a risky return, a more ambiguous asset for an agent with smooth ambiguity preferences corresponds to one with a higher variance for the second-order belief regarding the conditional mean.

[15] As in Table 4, the case with $\gamma = 0.5, \alpha = 5$ yields very similar results, which are avaibale upon request.

approaching to values closer to 1 (see Figure B.1 in Appendix B), optimal investment and hours choices become less variable overall. On the other hand, the variability of output and consumption tends to increase with greater information if agents know with more certainty the nature of the latent process generating the observations, as does the difference in the consumption and output responses across the high and low persistence processes. This latter finding occurs because with greater information, agents increase the responsiveness of their consumption if the process generating the observations is the high persistence process because they know that good times are likely to last longer relative to the situation with the low persistence process. Put differently, the precautionary saving motive becomes lessened under greater information or lower ambiguity. Thus, while the cyclicality of investment and hours fall with greater information, we observe an increase in the the responsiveness of output and consumption across the business cycle. On the other hand, the co-movement among the different series increases with greater informativeness or less ambiguity about the underlying TFP process across all parameter configurations as well as the nature of the TFP process that is being simulated.

The reason for these effects is that the standard deviations for the shocks to the unknown cyclical component to technological growth $x_{k,t}$, namely, σ_k, for $k = l, h$ assumed in Table 5 with informative priors are substantially larger than the standard deviations for the same shocks assumed in Table 4. Thus, in the versions of the model with more informative priors in the estimation of the temporary component of the underlying TFP process, as in Panels B and E of Table 5, agents are better able to infer the unknown temporary component of technology growth and to act on this knowledge to smooth their optimal investment and hours choices. Thus, it appears that the *informativeness* of the signals (improved signal-to-noise ratio) depends as much on their assumed standard deviations, as it does on knowledge of the exact process that is driving the behavior of the x_{kt} for $k = l, h$. These conclusions are, in general, valid when the technology process is drawn from the high persistence/low variability or the low persistence/high variability technology process. Finally, when agents are better able to infer the time series behavior of the latent process x_{kt} due to the more informative priors that used in the estimation of the unknown technology growth process, agents, in general, choose to *decrease* their precautionary saving, which also reduces the average level of the investment-output ratio.

Table 5: Simulation Results: Informative Priors about the Persistence/Variability of the Unobserved Component of TFP

	Panel B					Panel E				
	Simulations conditional on $\rho_k = 0.85$									
	Standard deviations									
	y	c	i	h	p	y	c	i	h	p
	1.388	0.653	6.057	0.661	0.84	1.58	0.819	5.944	.613	1.042
	Correlations									
	y	c	i	h	p	y	c	i	h	p
y	1.000	0.764	0.94	0.904	0.942	1.000	0.903	0.953	0.922	0.974
c		1.000	0.503	0.416	0.936		1.000	0.737	0.666	0.977
i			1.000	0.993	0.773			1.000	0.991	0.863
h				1.000	0.708				1.000	0.81
p					1.000					1.000
i/y					0.170					0.169
	Simulations conditional on $\rho_k = 0.30$									
	Standard deviations									
	y	c	i	h	p	y	c	i	h	p
	1.327	0.589	6.014	0.668	0.77	1.378	0.627	6.031	0.659	0.821
	Correlations									
	y	c	i	h	p	y	c	i	h	p
y	1.000	0.714	0.943	0.91	0.933	1.000	0.768	0.947	0.914	0.945
c		1.000	0.445	0.361	0.918		1.000	0.526	0.444	0.936
i			1.000	0.994	0.762			1.000	0.994	0.791
h				1.000	0.701				1.000	0.73
p					1.000					1.00
i/y					0.172					0.170

Note: The model is simulated based on the decision rules for the main model with ambiguity where the agent cannot distinguish perfectly between two processes with persistence $\rho_l = 0.30$ and $\rho_h = 0.85$. In Panel B, the parameters of the technology shock process are set so that $\rho_h = 0.85$, $\sigma_{A_h} = 0.00936$, $\sigma_{x_h} = 0.00139$ for the high persistence process and $\rho_l = 0.30$, $\sigma_{A_l} = 0.00905$, $\sigma_{x_l} = 0.00237$ for the low persistence process. In Panel E, the parameters of the technology shock process are set so that $\bar{g}_h = 0.00469$ and $\rho_h = 0.85$, $\sigma_{A_h} = 0.00915$, $\sigma_{x_h} = 0.00283$ for the high persistence process and $\rho_l = 0.30$, $\sigma_{A_l} = 0.00819$, $\sigma_{x_l} = 0.00467$ for the low persistence process.

In work that is related to ours, Bidder and Smith (2012) examine a real business cycle model with multiplier preferences which reflect the agent's desire for robust policies following Hansen and Sargent (2008). A further feature of this model is that shocks to the technology process display time varying volatility. In this case, the optimization problem of the agent takes the form of a two-player zero sum game between the robust agent (the maximizer) and a metaphorical 'evil' agent ('the minimizer') such that the agent seeks to minimize the distributions that twist continuation values towards outcomes that are painful to the agent. This latter term is represented by the conditional relative entropy associated with the twisted distribution. The solution of the minimization problem in a recursive representation of the problem yields a version of the log-exponential representation of the smooth ambiguity model that we discussed earlier. Bidder and Smith (2012) use perturbation methods to solve the model and to obtain approximations to the value function and other equilibrium objects. They then use sampling methods to generate the solution under the worst case joint distribution which arises endogenously and depends on draws of sequences over all variables in the economy. They find among other results that the model incorporating only shocks to time-varying volatility which is the source of mis-specification that leads to endogenous pessimism is able to explain 1%-16% of the variability of such series as output, consumption, investment and hours compared to the full stochastic model that contains both volatility and technology shocks.

To provide an analogous comparison regarding the impact of ambiguity, we compare the volatility of the endogenous series in the case of uninformative priors reported in Table 4 with those reported in Panels B and E of Table 5. We take the baseline case to be the case with uninformative priors in Table 4 and ask how lower ambiguity (defined in terms of the informative priors used in Table 5) affects the volatility and co-movement of the endogenous series. As in Bidder and Smith (2012), the distortions that arise under ambiguity and ambiguity averse preferences reflect distortions in all of the endogenous series, as the endogenous tilting or distortion that we described in Section 3.1.1 is based on the distortion function ξ_{kt} for $k = l, h$.

A straightforward calculation shows that the volatility of investment decreases by around 5.4% (7.2%) when agents have more information about the nature of the latent process driving productivity as described by Panels B (E) of Table 5 compared to the case with uninformative priors in Table 4 when the high persistence process is the correct model of TFP growth and by 4.8% (4.5%) if the low persistence process is the correct model. Likewise, the volatility of hours decreases by 7.6% (14.3%) when agents have more information about the nature of the latent process driving productivity as described in Panels B (E) of Table 5 compared to the case with uninformative priors in Table 4 if the high persistence process is the correct model of TFP growth and by 5.4% (6.7%) if the low persistence process is the correct model. The effect on output and consumption is to increase the difference in the consumption and output responses across the high and low persistence processes. We also find that the co-movement of all series increase when agents have more informative or less ambiguous signals.

Jaimovich and Rebelo (2009) obtain similar effects by considering the informativeness of news shocks in a Real Business Cycle model with so-called GHH preferences that reduce the wealth effect on hours worked. Specifically, they find that more informative signals about future investment-specific technological shocks tend to reduce the variability of the output, consumption, investment and hours series while increasing the co-movement among them. The difference between news shocks and ambiguity is that the former is typically followed by a realization of the shock itself whereas ambiguity here refers to the strength of the signal-noise ratio in thr estimated TFP process.

5 The risk-free rate

In this section, we characterize the behavior of the risk-free rate, and argue that it may serve as a further endogenous measure of ambiguity implied by our model. In the type of environments with full risk sharing that we have considered here, uncertainty aversion will tend to manifest itself in asset prices such as the risk-free rate (see Tallarini (2000) and Backus *et al.* (2015)).

Using the notation in Section 3, the (gross) risk-free rate R^f for the model with ambiguity aversion satisfies

$$1 = \beta R^f \Upsilon_t \left\{ \eta_t E_{\hat{x}_{l,t}} \left[\xi_t^{(l)} E_{x_{l,t}} \left(\exp(((1-\gamma)\nu - 1) g_{A_l,t+1}) \frac{\lambda_{t+1}^{(l)}}{\lambda_t} \right) \right] \right.$$
$$\left. + (1-\eta_t) E_{\hat{x}_{h,t}} \left[\xi_t^{(h)} E_{x_{h,t}} \left(\exp(((1-\gamma)\nu - 1) g_{A_h,t+1}) \frac{\lambda_{t+1}^{(h)}}{\lambda_t} \right) \right] \right\}. \quad (5.1)$$

where Υ_t and ξ_t are defined by equations (3.7) and (3.6) and $\lambda_t = \nu(\hat{c}_t^{\nu} l_t^{1-\nu})^{-\gamma}(\hat{c}_t/l_t)^{\nu-1}$.

In what follows, we derive a log-linear approximation to the risk-free interest rate under the assumptions in Collard *et al.* (2018), first, by considering the case with known persistence ($\eta_t = 0$)

for the growth rate of the TFP process and second, by treating the distorted or "as if" posterior $\tilde{\mu}_t \equiv \xi_t(x_t) \otimes N(\hat{x}_t, \Omega)$, where the distortion arises from the role of ambiguity aversion, as a normal density with variance Ω but a different mean, $\tilde{x}_t$. Using these assumptions, let $E_t \equiv E_{\hat{x}_t} E_{x_t}$ and $\tilde{E}_t \equiv E_{\tilde{\mu}_t} E_{x_t} \equiv E_{\tilde{x}_t} E_{x_t}$. Also, $\widetilde{Var}_t(x_t) = Var_{\tilde{x}_t}(x_t) = \Omega$ and $Var_t(x_t) = Var_{\hat{x}_t}(x_t) = \Omega$ and all ϵ terms have expectation zero under both $\tilde{E}_t$ and E_t since the terms have expectation zero conditional on x_t.[16]

Under these assumptions, the expression for the risk-free rate becomes

$$\begin{aligned}
1 &= \beta R^f \Upsilon_t \tilde{E}_t \left[\exp\left(\log(\lambda_{t+1}/\lambda_t) + ((1-\gamma)\nu - 1)(\bar{g} + \rho x_t + \sigma_x \epsilon_{x,t+1} + \sigma_A \epsilon_{A,t+1})\right)\right] \\
&= \beta R^f \Upsilon_t \exp\left[\tilde{E}_t\left(\log(\lambda_{t+1}/\lambda_t)\right) + \frac{\widetilde{Var}_t\left(\log(\lambda_{t+1}/\lambda_t)\right)}{2} + ((1-\gamma)\nu - 1)(\bar{g} + \rho\tilde{x}_t) \right. \\
&\quad \left. + \frac{((1-\gamma)\nu - 1)^2}{2}(\sigma_x^2 + \sigma_A^2) + \frac{((1-\gamma)\nu - 1)^2 \rho^2}{2} \widetilde{Var}_t(x_t)\right],
\end{aligned}$$

which implies

$$\begin{aligned}
r^f &= -\log(\beta) - \log(\Upsilon_t) - \left[\tilde{E}_t\left(\log(\lambda_{t+1}/\lambda_t)\right) + \frac{\widetilde{Var}_t\left(\log(\lambda_{t+1}/\lambda_t)\right)}{2} + (((1-\gamma)\nu) - 1)(\bar{g} + \rho\tilde{x}_t) \right. \\
&\quad \left. + \frac{((1-\gamma)\nu - 1)^2}{2}(\sigma_x^2 + \sigma_A^2) + \frac{((1-\gamma)\nu - 1)^2 \rho^2}{2} \widetilde{Var}_t(x_t)\right],
\end{aligned}$$

[16] In these expressions, the mean and variance of the distorted or "as if" posterior distribution which allow for uncertainty about the persistence parameter, and incorporate the probability that an agent places on the probability of the low versus the high persistence process are given as follows:

$$\tilde{x}_t = \eta_t \int_{-\infty}^{\infty} (x_{l,t}) \xi_t^{(l)} dF(x_{l,t}) dx_{l,t} + (1-\eta_t) \int_{-\infty}^{\infty} (x_{h,t}) \xi_t^{(h)} dF(x_{h,t}) dx_{h,t}, \tag{5.2}$$

and

$$\widetilde{Var}_t(x_t) = \eta_t \int_{-\infty}^{\infty} (x_{l,t}^2) \xi_t^{(l)} dF(x_{l,t}) dx_{l,t} + (1-\eta_t) \int_{-\infty}^{\infty} (x_{h,t}^2) \xi_t^{(h)} dF(x_{h,t}) dx_{h,t} - \tilde{x}_t^2. \tag{5.3}$$

where $r^f = \log(R^f)$. Now

$$\log(\Upsilon_t) = \log \tilde{E}_t \left(\hat{J}(\hat{k}_{t+1}, \hat{x}_{t+1}) \exp(\nu(1-\gamma) g_{A,t+1})\right)^{-\alpha}$$
$$+\frac{\alpha}{1-\alpha} \log \tilde{E}_t \left(\hat{J}(\hat{k}_{t+1}, \hat{x}_{t+1}) \exp(\nu(1-\gamma) g_{A,t+1})\right)^{1-\alpha}$$
$$= -\alpha \tilde{E}_t \left(\log(\hat{J}(\hat{k}_{t+1}, \hat{x}_{t+1})) + (\nu(1-\gamma) g_{A,t+1})\right)$$
$$+\alpha \tilde{E}_t \left(\log(\hat{J}(\hat{k}_{t+1}, \hat{x}_{t+1})) + (\nu(1-\gamma) g_{A,t+1})\right) + \text{variance terms},$$

so that the direct effect of α on the risk-free interest rate cancels out as before. Based on these rules, we rewrite the risk-free rate as

$$r^f = -\log(\beta) - \left[\tilde{E}_t \left(\log(\lambda_{t+1}/\lambda_t)\right) + \frac{\widetilde{Var}_t \left(\log(\lambda_{t+1}/\lambda_t)\right)}{2} + ((1-\gamma)\nu - 1)(\bar{g} + \rho \tilde{x}_t)\right.$$
$$\left. + \frac{((1-\gamma)\nu - 1)^2}{2}(\sigma_x^2 + \sigma_A^2) + \frac{((1-\gamma)\nu - 1)^2 \rho^2}{2} \widetilde{Var}_t(x_t)\right] + \text{extra variance terms}$$

To understand the impact of ambiguity aversion on the risk-free rate, we note that the log-linear rule for the risk-free interest implies that r^f depends positively on the distorted posterior mean, $\tilde{x}_t$, and negatively on the distorted posterior variance, $\widetilde{Var}_t(x_t)$.[17] From the definition of ξ_t, since the distorted posterior mean declines with increases in α due to the endogenous tilting of beliefs under ambiguity aversion while the distorted posterior variance increases with α, we expect that an increase in ambiguity aversion will tend to reduce the risk-free rate. This may be interpreted as reflecting the increased demand for risk-free assets in an environment with endogenous doubt and pessimism.

The risk-free rate also plays a key role in Collard *et al.* (2018)'s analysis of the historical equity premium. They note that "ambiguity aversion gets the first moment of equity premium right by holding down the risk-free rate while affecting the risky rate only very marginally."[18] Tallarini (2000) notes that generating an equity premium that is consistent with the data in a

[17]Notice that the term $((1-\gamma)\nu - 1)$ is negative for all values of $0 \leq \gamma \leq 1$ since $0 < \nu < 1$.

[18]In their case, a term similar to Υ_t is equal to unity, and the ratio of the Lagrange multipliers depends only on consumption growth, which they take as exogenous. In our case, the term λ_{t+1}/λ_t depends on consumption and leisure choices, which are determined as functions of the transformed capital stock and the evolution of agents' beliefs.

production economy requires the addition of frictions of adjustment costs or other frictions that will generate variation in the price of capital. While we have considered the presence of a friction such as investment irreversibility, our focus is not directly on accounting for various asset pricing phenomena considered in the literature.

5.1 Simulations

In Table 6, we report the average interest rate and the mean and standard deviation of "as if" or distorted beliefs that are implied by the smooth ambiguity model. We simulate 100 different economies of 275 observations each corresponding to the sample period 1947:I-2015:IV, and report values after discarding the burn-in sample of 1947:I-1977:IV. As in our other simulations exercises, we implement the simulations under the assumption that the TFP shocks are drawn from the high persistence/low variability versus low persistence/high variability processes. However, since the values of the average interest rate did not change significantly across the processes with high or low persistence, we only report the values for the high persistence process. We also consider the case of $\gamma = 0.5, \alpha = 0.8$ throughout the simulations.[19]

Studies such as Bloom *et al.* (2018) or Gilchrist *et al.* (2014) construct alternative measures of uncertainty through the aggregation of individually observed quantities. Specifically, Bloom *et al.* (2018) model uncertainty as the volatility of establishment-level measures of total factor productivity in US 4-digit industries while Gilchrist *et al.* (2014) construct an observed measure of uncertainty as time-varying idiosyncratic volatility in firms' stock returns. To understand the relation between their proposed measures of uncertainty and ambiguity, we present results for the risk-free rate using sectoral TFP data available from Bloom *et al.* (2018), and compare our implied ambiguity measures with their measures of uncertainty at the sectoral level.[20] For this purpose, we

[19] It is also possible to use a higher value of $\beta = 0.9926$ (see, for example, Christiano and Eichenabum (1992)), which will tend to reduce the risk-free rate further but our focus here is on understanding the role of ambiguity.

[20] Bloom *et al.* (2018) use the Census of Manufacturers and the Annual Survey of Manufacturers to construct an establishment-level panel data set. To generate their TFP uncertainty measures, they define value-added TFP for each establishment j in industry i at time t as $\log(z_{j,i,t} = \log(v_{j,i,t}) - \alpha^S_{i,t}\log(k^S_{j,i,t}) - \alpha^E_{j,i,t}\log(k^E_{j,i,t}) - \alpha^N_{i,t}\log(n_{j,i,t})$, where $v_{j,i,t}$ denotes value-added, $k^S_{j,i,t}$ is the stock of structures, $k^E_{j,i,t}$ is the stock of equipment, and $n_{j,i,t}$ is total hours worked, and $\alpha^k_{i,t}, k = S, E, N$ are the cost shares of the different inputs. TFP shocks $e_{j,i,t}$ are then defined as the residual from establishment-level log TFP from a first-order autoregressive equation with time and industry dummies. At the industry level, they then use interquartile range (IQR) and the weighted mean of absolute values

re-estimate TFP growth processes using annual sectoral TFP measures provided by Bloom *et al.* (2018) for 4-digit manufacturing industries indicated in Table 6.[21]

Table 6 reports the distorted mean, the distorted standard deviation, the implied risk-free rate for the different models using aggregate and sectoral TFP data together with the uncertainty measure computed by Bloom *et al.* (2018) for the relevant industries. This table shows that the real interest rate implied under the baseline specification as well as the one under the informative prior B are estimated to be nearly equal and large, reflecting the low variation in the distorted beliefs under these estimates. Hence, the notion of a "flight to safety" in response to the induced endogenous pessimism does not manifest itself in significantly low interest rates for the specifications based on aggregate TFP data.

Turning to the sectoral results, the results show that industries for which the distortions induced by ambiguity and ambiguity aversion are relatively large (as measured by the distorted mean and standard deviation of $\tilde{x}_t$) also tend to display lower values of the risk-free rate. These industries include Petroleum Mining (2911), Plastic Pipes (3084), Steel Wiredrawing, Steel Nails and Spikes (3315), Internal Combustion Engines (3519), and Motors and Generators (3621). We also observe that for some industries the uncertainty measures used by Bloom *et al.* (2018) correlate with our induced ambiguity measures. As an example, we find that for Petroleum Mining (2911), a high uncertainty measure is accompanied by evidence of significant ambiguity regarding the cyclical component of the underlying TFP process, as indicated by a high distorted standard deviation and a low risk-free rate. Given the volatile and unpredictable nature of technological developments in this industry such as fracking, one may rationalize the presence of high uncertainty and ambiguity for this sector. Conversely, for a low uncertainty industry such as Glass Containers, a relatively low distorted standard deviation leads to a moderately high interest rate. However, Steel Wiredrawing,

as uncertainty measures. See their Appendix A for further descriptions.

[21] In this section, we do not report the business cycle moments implied under the sectoral TFP estimates, though they are available upon request. Nevertheless, we choose industries for which the relative volatilities of output, consumption and investment are comparable to the aggregate TFP estimates, although the overall volatilities may differ across different industries. Another issue which we have not addressed when using the sectoral TFP estimates in the aggregate business cycle model is the existence of increasing returns to scale in production at the sectoral level, which would lead to model mis-specification. Hall (1988, 1990) argued for the existence of increasing returns and significant market power in many US industries. Subsequently, however, authors such as Basu and Fernald (1997) and Burnside (1996) suggest that US manufacturing industry essentially exhibits constant returns to scale.

Steel Nails and Spikes (3315) has a very high distorted standard deviation and a low implied risk-free rate but its uncertainty measure is considerably lower compared to Petroleum Mining. Evidently, the uncertainty measures reported by Bloom *et al.* (2018) and our induced ambiguity measures are indicative of alternative aspects of the external environment facing agents.

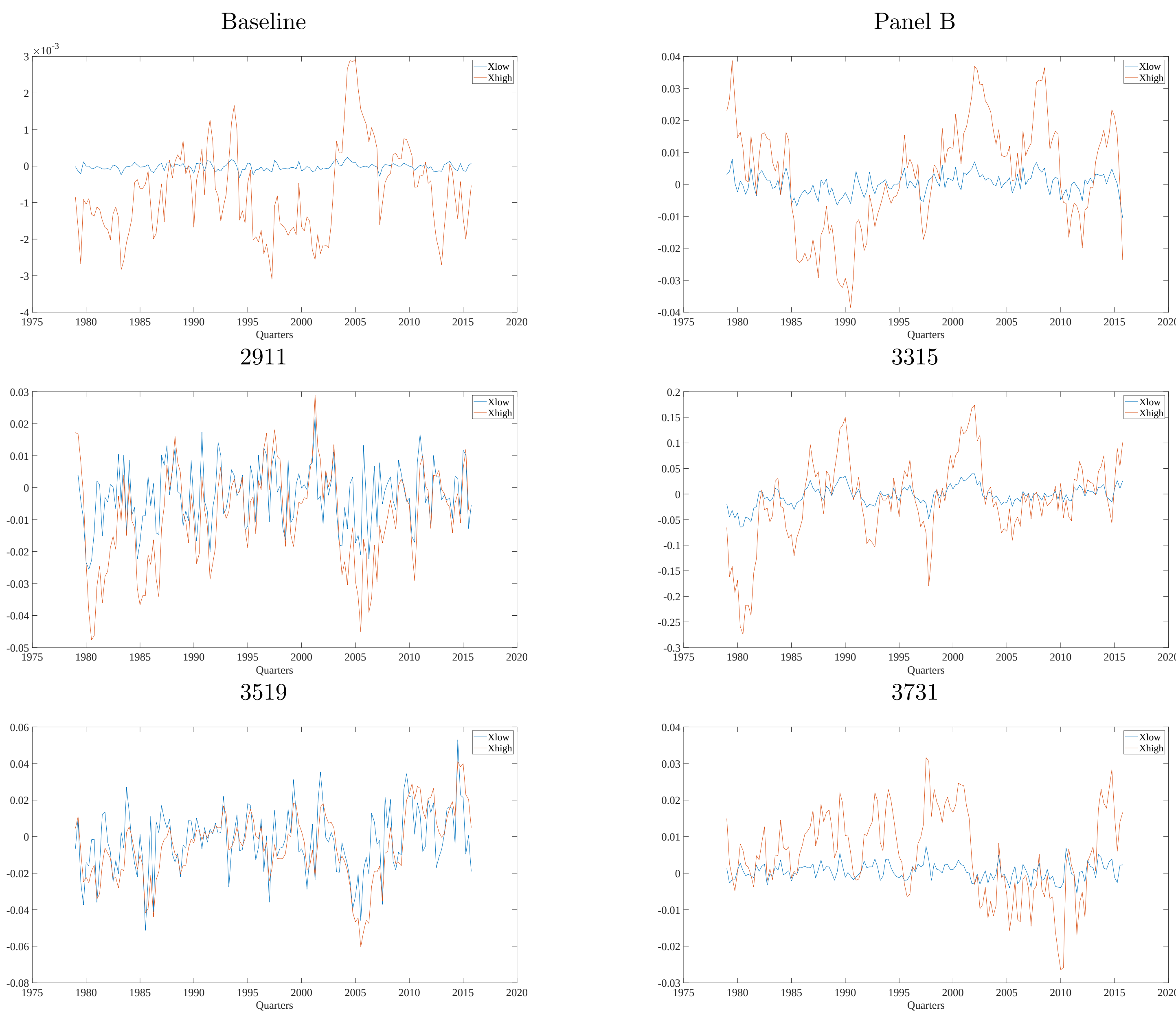
To further understand the reasons for these findings, Figure 5 displays the filtered means $\hat{x}_{lt}$ (in 'blue') and $\hat{x}_{ht}$ (in 'red') (in percentage terms) for the the low and high persistence processes for selected cases. These figures provide a way to understand the nature of ambiguity facing agents. An ambiguity-averse agent endogenously behaves as if the uncertainty is more persistent and severe following negative shocks than in normal times. Considering the baseline model, we observe some cyclical declines in $\hat{x}_{ht}$ but the magnitude of variation in both $\hat{x}_{ht}$ and $\hat{x}_{lt}$ is minor, implying little information in the aggregate TFP series when uninformative priors are used about the underlying latent cyclical component of TFP. On the other hand Figure 5 shows that for Petroleum Mining, the mean of the high persistence process tends display significant drops, and to fall below the mean of the low persistence process. Since ambiguity-averse agents forecast TFP growth by putting more weight on the "worst case persistence", in situations with negative shocks where $\hat{x}_{ht} < \hat{x}_{lt}$, the worst case persistence is ρ_h, suggesting that the economy will remain in the bad state for a long time. This tends to increase the endogenous distortions and to lead to greater ambiguity compared to situations for which $\hat{x}_{ht}$ does not fall systematically below $\hat{x}_{lt}$. On the other hand, when the filtered mean of the high persistence process is above the filtered mean of the low persistence process, the worst case is that the persistence is ρ_l, suggesting that the good times will not last that long. Shipbuilding and Repairing (3731) has a relatively high uncertainty index according to measures comouted by Bloom *et al.* (2018) but the filtered mean for the high persistence process rarely falls below the filtered mean for the low persistence process except for the period preceding the global financial crisis, implying the absence of pessimist beliefs and hence, a relatively high risk-free rate. Finally, Steel Wiredrawing, Steel Nails and Spikes (3315) displays episodes when $\hat{x}_{ht}$ falls very far below $\hat{x}_{lt}$ (though the converse is also observed), implying the exacerbation of uncertainty through endogenous belief mechanisms.

Table 6: Simulation Results for the Risk-free Rate, Quarterly Rate

Model	Industry name	Risk-free rate r^f (in %)	$\tilde{X}$ (in %)	$SD(\tilde{X})$ (in %)	Uncertainty measure
Baseline model	–	1.55	-6.00e-05	0.063	–
Panel B	–	1.54	-2.43e-03	0.250	–
	Representative industries				
Industry 2421	Sawmills and planing mill, general	1.30	-2.96e-02	0.40	0.464
Industry 2869	Industrial organic chemicals	1.27	6.22e-03	0.23	0.529
Industry 2911	Petroleum mining	1.07	4.76e-04	0.41	0.650
Industry 3084	Plastic pipes	1.07	1.90e-03	0.31	0.424
Industry 3221	Glass containers	1.17	-6.07e-04	0.23	0.193
Industry 3315	Steel wiredrawing, steel nails and spikes	1.07	-8.00e-03	0.57	0.365
Industry 3448	Fabricated metal buildings and components	1.28	3.82e-03	0.23	0.395
Industry 3519	Internal combustion engines	1.05	4.21e-03	0.39	0.426
Industry 3621	Motors and generators	1.11	-4.47e-03	0.35	0.351
Industry 3731	Ship building and repairing	1.61	2.63e-03	0.28	0.407

Note: The model is simulated based on the decision rules for the main model with ambiguity where the agent cannot distinguish perfectly between two processes with persistence $\rho_l = 0.30$ and $\rho_h = 0.85$. The riskfree interest rate is simulated for the full sample of 1947:i-2015:IV but the simulated values for the period 1947:I-1978:IV are discarded as part of the burn-in sample. The distorted means, $\tilde{X}$, and the distorted standard deviations, $SD(\tilde{X})$, are computed according to formulas in Section 5 and are evaluated in percentage terms, as is the risk-free rate. The uncertainty measure is taken from Bloom *et al.* (2018).

Figure 5: Filtered beliefs underlying the risk-free rate values - Aggregate TFP and Industry Results



6 Conclusion

This paper examines the cyclical dynamics of a Real Business Cycle model with ambiguity averse consumers and investment irreversibility using the smooth ambiguity model of Klibanoff *et al.* (2005, 2009). In this model, agents do not know which distribution the unobserved temporary component of TFP growth is coming from, and learn about it based on observations of current and past values of the TFP growth. We find that a suitably parametrized version of the Real Business Cycle model can match the standard business cycle moments studied in the literature, though consumption variability tends to be lower than observed in the data.

Using aggregate TFP data, we find that the model has the ability to generate significant effects of ambiguity and ambiguity aversion when the impact of ambiguity is modeled as greater informativeness regarding estimates of the underlying TFP process. In our analysis, we measure such informativeness through the use of priors with greater signal-to-noise ratios which enable the agent to discriminate better between the low and high persistent processes. Viewing this as a shock to the agent's "confidence" regarding the nature of the underlying TFP process, we find that greater confidence or lower ambiguity is associated with a decline in the precautionary savings motive, which leads to lower cyclical variability in investment and hours choices and greater co-movement in the key macroeconomic aggregates. On the other hand, the variability of output and consumption tends to increase with greater information if agents know with more certainty the nature of the latent process generating the observations, as does the difference in the consumption and output responses across the high and low persistence processes.

In contrast to earlier studies such as Ilut and Schneider (2014) or Bidder and Smith (2012), our analysis examines both aggregate and sectoral data to understand the quantitative impact of uncertainty on prices and quantities and emphasizes the information channel in the transmission of ambiguity and ambiguity aversion. In addition to examining business cycle moments, we show that ambiguity manifests itself through the risk-free rate. Specifically, the greater the endogenous distortions induced by ambiguity and ambiguity aversion, the lower is the risk-free rate in equilibrium. In this respect, we find that sectoral TFP data allow for the ability to differentiate the impact of ambiguity through the properties of the filtered and distorted beliefs that comprise the risk-free rate. Finally, we find that standard uncertainty measures do not always correlate with ambiguity measures based on similar data.

Undoubtedly, uncertainty and ambiguity are important factors deriving agents' decisions to work, to invest, and to consume. Even in an environment where the choices of agents can be characterized as the solution to a social planner's problem, we find that ambiguity can have sizable effects on endogenous quantities. On the other hand, introducing various forms of market incompleteness may work towards generating a greater role for ambiguity and ambiguity aversion. Second, an environment that considers individual heterogeneity may be more likely to capture the impact of such uncertainty/ambiguity aversion. A third issue has to do with the measurement of TFP shocks in order to gauge their influence on business cycles. As our results on the risk-free rate suggest, using more disaggregated measures of productivity changes at the firm or industry level in a general equilibrium economy with multiple sectors may provide more useful approaches to generating the impact of ambiguity and ambiguity aversion. We leave exploring these avenues for future work.

A Numerical solution approach

We now describe how to numerically solve the social planner's problem described in Section 3.1. Our task is we determine the function $\hat{J}(\hat{k}_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t)$ for all values of the normalized capital stock, $\hat{k}_t$, and for the variables specifying beliefs, $(\hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t)$. Unlike Collard *et al.* (2018) who consider an endowment economy, we also need to calculate the optimal investment and hours policies as part of the numerical solution for the indirect value function. Notice that the optimization routine needs to account for the inequality constraint on the choice of $\hat{\imath}_t$.[22] We use the method of value iteration with Chebyshev interpolation, which involves approximating the function $\hat{J}(\hat{k}_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t)$ by a parametric function whose coefficients are determined according to a minimum residual method; see Judd (1998)

[22] For details of the solution procedure, see Ju and Miao (2012), Jahan-Parvar and Liu (2014), Collard *et al.* (2018), and Liu and Zhang (2018), among others.

We begin by explicitly writing the expectations that appear in this formulation.

$$\hat{J}(\hat{k}_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) = \max_{\hat{c}_t, n_t, \hat{i}_t} \left\{ \frac{((\hat{c}_t{}^{\nu} l_t^{1-\nu})^{1-\gamma}}{1-\gamma} + \right.$$

$$\beta \left[\eta_t \left(\int_{-\infty}^{\infty} \left(\int\int_{-\infty}^{\infty} \hat{J}(\hat{k}_{t+1}^{(l)}, \hat{x}_{h,t+1}^{(l)}, \hat{x}_{l,t+1}^{(l)}, \eta_{t+1}^{(l)}) \exp(\nu(1-\gamma) g_{A_l,t+1}) dF(\varepsilon_{l,t+1}) \right)^{1-\alpha} dF(x_{l,t}) \right) + \right.$$

$$\left. \left. (1-\eta_t) \left(\int_{-\infty}^{\infty} \left(\int\int_{-\infty}^{\infty} \hat{J}(\hat{k}_{t+1}^{(h)}, \hat{x}_{h,t+1}^{(h)}, \hat{x}_{l,t+1}^{(h)}, \eta_{t+1}^{(h)}) \exp(\nu(1-\gamma) g_{A_h,t+1}) dF(\varepsilon_{h,t+1}) \right)^{1-\alpha} dF(x_{h,t}) \right) \right]^{\frac{1}{1-\alpha}} \right\}.$$

subject to $\hat{c}_t + \hat{i}_t \leq \hat{k}_t^a n_t^{1-a}$, $\exp(g_{A,t+1})\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \hat{i}_t$, $l_t + n_t \leq 1$, and $\hat{i}_t \geq 0$. Here $\hat{x}_{h,t+1}^{(k)}, \hat{x}_{l,t+1}^{(k)}, \eta_{t+1}^{(k)}$ are functions of $\varepsilon_{k,t+1} = (\epsilon_{x_k,t+1}, \epsilon_{A_k,t+1})', k = l, h$, which is a 2 by 1 vector standard normal shocks and $\eta_{t+1}^{(l)}$ is the posterior probability at time $t+1$ that the model with ρ_l is the true data generating process. $F(\varepsilon_{k,t+1}), k = l, h$ are both bivariate independent standard normal distributions while $F(x_{k,t}), k = l, h$ is a normal distribution with mean $\hat{x}_{k,t}$ and variance Ω_k, which is defined below.

The updates for $\hat{x}_{k,t+1}^{(i)}$ are obtained using the Kalman filter algorithm as follows:

$$\begin{aligned}
\hat{x}_{l,t+1}^{(l)}(\varepsilon_{l,t+1}) &= \rho_l \hat{x}_{l,t} + K_l \nu_{l,t+1}^{(l)}, \\
\hat{x}_{h,t+1}^{(l)}(\varepsilon_{l,t+1}) &= \rho_h \hat{x}_{h,t} + K_h \nu_{h,t+1}^{(l)}, \\
\hat{x}_{l,t+1}^{(h)}(\varepsilon_{h,t+1}) &= \rho_l \hat{x}_{l,t} + K_l \nu_{l,t+1}^{(h)}, \\
\hat{x}_{h,t+1}^{(h)}(\varepsilon_{h,t+1}) &= \rho_h \hat{x}_{h,t} + K_h \nu_{h,t+1}^{(h)},
\end{aligned}$$

where $\nu_{k,t+1}^{(i)}, (i) = l, h, k = l, h$ are the "surprises". For example, when the DGP is $(i) = l$ and the filter uses $\rho_k, k = h$, the surprise is defined as

$$\begin{aligned}
\nu_{h,t+1}^{(l)} &= g_{A_l,t+1} - \bar{g} - \rho_h \hat{x}_{h,t} \\
&= \bar{g} - \bar{g} + \rho_l x_{l,t} - \rho_h \hat{x}_{h,t} + \sigma_{x_l}\epsilon_{x_l,t+1} + \sigma_{A_l}\epsilon_{A_l,t+1} \\
&= \rho_l x_{l,t} - \rho_h \hat{x}_{h,t} + \sigma_{x_l}\epsilon_{x_l,t+1} + \sigma_{A_l}\epsilon_{A_l,t+1}.
\end{aligned}$$

The Kalman gain parameters, $K_k, k = l, h$, depending on whether the low or high persistence model is assumed to be the true model, respectively, are

$$K_k = \rho_k \Omega_k f_k^{-1}, \quad f_k = \Omega_k + \sigma_{A_k}^2,$$

where $f_k = E[(g_{A_k,t+1} - E(g_{A_k,t+1}))^2 | g_{A,1}, \ldots, g_{A,t}]$, and $\Omega_k = E[(x_{k,t+1} - \hat{x}_{k,t+1})^2 | g_{A,1}, \ldots, g_{A,t}]$, $k = l, h$ are defined as the solution to $\Omega_k = \rho_k^2 \Omega_k - \rho_k^2 \Omega_k^2 f_k^{-1} + \sigma_{x_k}^2$.[23] The Bayes update of η_t is obtained as follows:

$$\begin{aligned}
\eta_{t+1}^{(l)}(\varepsilon_{l,t+1}) &= \frac{\eta_t L(\nu_{l,t+1}^{(l)}, f_l)}{\eta_t L(\nu_{l,t+1}^{(l)}, f_l) + (1-\eta_t) L(\nu_{h,t+1}^{(l)}, f_h)}, \\
\eta_{t+1}^{(h)}(\varepsilon_{h,t+1}) &= \frac{\eta_t L(\nu_{l,t+1}^{(h)}, f_l)}{\eta_t L(\nu_{l,t+1}^{(h)}, f_l) + (1-\eta_t) L(\nu_{h,t+1}^{(h)}, f_h)},
\end{aligned}$$

where the likelihood is

$$L(\nu_{j,t+1}^{(i)}, f_j) = \frac{1}{2\pi\sqrt{f_j}} \exp\left(-\frac{(\nu_{j,t+1}^{(i)})^2}{2 f_j} \right).$$

A.1 The numerical algorithm

Following the recent literature (see, e.g. Richter *et al.* (2014), we approximate the indirect value function $\hat{J}(\hat{k}_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t)$ by a parametric function of the form

$$\Phi(X_t) = \exp\left(\sum_{i_k, i_i, i_h, i_\eta \in \Upsilon} c_{i_k, i_l, i_h, i_\eta} T_{i_k}(\hat{k}_t) T_{i_l}(\hat{x}_{l,t}) T_{i_h}(\hat{x}_{h,t}) T_{i_\eta}(\hat{\eta}_t) \right), \tag{A.1}$$

[23] These results are obtained by applying the Kalman filter algorithm to the state and measurement equations as

$$\begin{aligned}
x_{k,t+1} &= \rho_k x_{k,t} + \sigma_{x_k} \epsilon_{x_k,t+1}, \quad k = l, h, \\
g_{A_k,t+1} - \bar{g} &= x_{k,t+1} + \sigma_{A_k} \epsilon_{A_k,t+1}, \quad k = l, h.
\end{aligned}$$

The expression for the gain parameters K_k and the variances of the filtered estimates of $x_{k,t}$ denoted Ω_k are obtained as a direct application of the Kalman filter algorithm. A similar application of the Kalman filter yields the expressions for $x_{k,t+1}^{(i)}$ and the surprises $\nu_{k,t+1}^{(i)}$ for $k = l, h$ and $i = l, h$. See Anderson and Moore (1979), Ch. 3.

where $X_t = (\hat{k}_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \hat{\eta}_t)$ denotes the vector of state variables. Notice that in the full information case, the vector of state variables is given by $X_t = (\hat{k}_t, x_t)$ where x_t is temporary component of the technology shock. In this expression, the set of indices is defined as $\Upsilon = \{i_z = 1, \ldots, n_z; z \in \{k, l, h, \eta\} | i_k + i_l + i_h + i_\eta \leq \max(n_k, n_l, n_h, n_\eta)\}$. This definition assumes that we are considering a complete basis of polynomials. In this expression $T_n(\cdot)$ is the Chebyshev functions $T(\cdot)$ are defined as the unique polynomials satisfying $T_n(x) = \cos(n \arccos x)$, or equivalently, $T_n(\cos(\nu)) = \cos(n\nu)$. Using a recursive formulation, we have $T_0(x) = 1, T_1(x) = x$ and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$. Using the definition that $\cos\left((2k+1)\frac{\pi}{2}\right) = 0$, it is possible to show that the roots of T_n are

$$x_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, \ldots, n.$$

The roots of the Chebyshev polynomial are also called Chebyshev nodes because they are used as nodes in polynomial interpolation. The orders of the Chebyshev functions $T_{i_z,n+1}(x)$ are set as $(n_k, n_{x_h}, n_{x_l}, n_\eta) = (4, 2, 2, 2)$, and 8 nodes each are used to evaluate the Chebyshev functions, yielding a total of 4096 nodes

Next, define the functions

$$\begin{aligned} \kappa_l(x_{l,t}) &= \left(E_{x_{l,t}} \Phi(X_{t+1}^{(l)}) \exp((1-\gamma) g_{A_l,t+1})\right)^{1-\alpha} \\ &= \left(\int_\infty^\infty \int_\infty^\infty \Phi(X_{t+1}^{(l)}) \exp((1-\gamma) g_{A_l,t+1}) dF(\varepsilon_{l,t+1})\right)^{1-\alpha} \end{aligned} \tag{A.2}$$

and

$$\begin{aligned} \kappa_h(x_{h,t}) &= \left(E_{x_{l,t}} \Phi(X_{t+1}^{(h)}) \exp((1-\gamma) g_{A_h,t+1})\right)^{1-\alpha} \\ &= \left(\int_\infty^\infty \int_\infty^\infty \Phi(X_{t+1}^{(h)}) \exp((1-\gamma) g_{A_h,t+1}) dF(\varepsilon_{h,t+1})\right)^{1-\alpha} \end{aligned} \tag{A.3}$$

Notice that the indirect value function can be expressed as

$$\hat{J}(\hat{k}_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) = \max_{\hat{i}_t} \left\{ \frac{(\hat{k}_t^a - \hat{i}_t)^{1-\gamma}}{1-\gamma} + \left[\eta_t E_{\hat{x}_{l,t}} \kappa_l(x_{l,t}) + (1-\eta_t) E_{\hat{x}_{h,t}} \kappa_h(x_{h,t})\right]^{\frac{1}{1-\alpha}} \right\}.$$

To be able to evaluate the value function, we also need to approximate the integrals that appear in this expression using numerical integration procedures.

- Gauss-Hermite quadrature: In the case of uni-dimensional integrals (as in the outer integral involved in the computation of expectations such as (A.4)), a Gauss Hermitian quadrature approach. Specifically, consider the expectations of the form $E_{\hat{x}_{k,t}}[K_k(x_{k,t})], k = l, h$. Since $x_{k,t}$ is distributed as normal with mean $\hat{x}_{k,t}$ and variance Ω_k, we apply a change of variables $z_{k,t} = (x_{k,t} - \hat{x}_{k,t})/\sqrt{2\Omega_k}$ to write the Guass-Hermite quadrature rule as

$$
\begin{aligned}
E_{\hat{x}_{k,t}}[\kappa_k(x_{k,t})] &= \pi^{-1/2} \int_{-\infty}^{\infty} \kappa_k(\sqrt{2\Omega_k}(z_{l,t} + \hat{x}_{k,t}))dF(x_{k,t}), \\
&\approx \pi^{-1/2} \sum_{i=1}^{n} \omega_i \kappa_k(\sqrt{2\Omega_k}(z_{k,t} + \hat{x}_{k,t}))
\end{aligned}
$$

  where $\omega_i = 2^{n+1} n! \sqrt{\pi}[H_{n+1}(x_i)]^{-2}$ and H_{n+1} is the Hermite polynomial of order n.

- The monomial approach: In the case of multi-dimensional integrals such as (A.2) or (A.3), Collard *et al.* (2018) use a monomial approach; see Judd (1998), p. 271-276) with 5 degree rule for an integrand on an unbounded range weighted by the standard normal. Specifically, we approximate the multi-dimensional integral

$$
\int_{\infty}^{\infty} \int_{\infty}^{\infty} \Phi(X_{t+1}^{(k)}) \exp((1-\gamma) g_{A_k,t+1})) dF(\varepsilon_{k,t+1})
$$

  by a 5 degree rule using $2d + 1$ points with $d = 2$ as

$$
a_0 \Phi(0) + a_1 \sum_{i=1}^{d} (\Phi(re^i) + \Phi(-re^i)) + a_2 \sum_{i=1}^{d-1} \sum_{j=1}^{d} \left(\Phi(\pm se^i + \pm se^j) \right), \tag{A.4}
$$

where e^i denotes the ith column vector of the identity matrix of order $d = 2$, and

$$r = \sqrt{1 + \frac{1}{2}d}, \quad s = \sqrt{\frac{1}{2} + \frac{d}{4}}, \quad v = \pi^{d/2},$$
$$a_0 = \frac{2}{d+2}v, \quad a_1 = \frac{4-d}{2(d+2)^2}, \quad a_2 = \frac{v}{(d+2)^2}.$$

Suppose we obtain an approximation to the indirect value function at the τ'th iteration using these steps. This will be based on the Chebyshev coefficients at the τ'th stage of the algorithm, $\mathbf{c}^\tau$. Denote the approximation obtained by using these coefficients by $\hat{J}^{(\tau)}(X_t)$. Also define the vector of future state variables by $X_{t+1}^{(k)} = (\hat{k}_{t+1}^{(k)}, \hat{x}_{h,t+1}^{(k)}, \hat{x}_{l,t+1}^{(k)}, \eta_{t+1}^{(l)}), k = l, h$. The value function and optimal investment policy functions, $\hat{J}^*(X_t)$ and $\hat{i}_t^* = g(X_t)$, are obtained as the solution to

$$\hat{J}^*(X_t) = \max_{\hat{i}_t} \left\{ \frac{((\hat{k}_t^a n_t^{1-a} - \hat{i}_t)^\nu n_t^{1-\nu})^{1-\gamma}}{1-\gamma} + \beta \left[\eta_t E_{\hat{x}_{l,t}} \left(E_{x_{l,t}} J^{(\tau)}(X_{t+1}^{(l)}) \exp(\nu(1-\gamma) g_{A_l,t+1}) \right)^{1-\alpha} \right.\right.$$
$$\left.\left. + (1-\eta_t) E_{\hat{x}_{h,t}} \left(E_{x_{h,t}} J^{(\tau)}(X_{t+1}^{(h)}) \exp(\nu(1-\gamma) g_{A_h,t+1}) \right)^{1-\alpha} \right]^{\frac{1}{1-\alpha}} \right\}.$$

subject to $\exp(g_{A,t+1})\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \hat{i}_t$ and $\hat{i}_t \geq 0$. Now, at the end of the τ'th iteration, the new value function $\hat{J}^*(X_t)$ is used to update the coefficients of the Chebyshev polynomials, and to obtain $\hat{J}^{(\tau+1)}(X_t)$ as we describe below.

Denote by $\mathbf{c}^\tau, \mathbf{c}^{\tau+1}$ as the set of coefficients entering (A.1) at the τ'th and $\tau + 1$'th stages, respectively. We determine the set of coefficients, $\mathbf{c}^{\tau+1}$ at the $\tau + 1$'th stage, using a minimum weighted residual method. Recall that the indirect value function depends on the coefficients from the τ'th stage as $J^*(X_t; \mathbf{c}^\tau)$ while the new approximation for the indirect value function depends on $\mathbf{c}^{\tau+1}$. The residual function associated with the new set of Chebyshev coefficients is given by $\mathcal{R}(X_t; \mathbf{c}^{\tau+1})$, where $\mathcal{R}(X_t; \mathbf{c}^{\tau+1}) = \Phi(X_t; \mathbf{c}^{\tau+1}) - \hat{J}^*(X_t; \mathbf{c}^\tau)$. This involves solving the problem

$$\min_{\mathbf{c}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\Phi(X_t; \mathbf{c}^{\tau+1}) - \hat{J}^*(X_t; \mathbf{c}^\tau))^2 \omega(X_t) dX_t, ,$$

where $\omega(X)$ is a multi-dimensional weighting function. The first-order conditions for this problem with respect to elements of $c_{i_z}, i_z \in \Upsilon$ are

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\Phi(X_t;\mathbf{c}^{\tau+1}) - \hat{J}^*(X_t;\mathbf{c}^{\tau}))T_{i_k}(\hat{k}_t)T_{i_l}(\hat{x}_{l,t})T_{i_h}(\hat{x}_{h,t})T_{i_\eta}(\eta_t)\omega(X_t)dX_t = 0. \quad \text{(A.5)}$$

If we assume that the weighting function is the product of the weights for $z \in \{k, l, h, \eta\}$ defined as

$$\omega_{i_z}(y_z) = \frac{T_{i_z}(y_z)}{\sqrt{1 - y_z^2}}, \quad i_z = 1, \ldots, n_z, i_z \in \Upsilon,$$

where $y \in \{\hat{k}, \hat{x}_l, \hat{x}_h, \eta\}$), the integral in the orthogonality conditions can be solved using Gauss-Chebyshev quadrature. For integrals of this form, the quadrature nodes and the (constant) quadrature weights are given by

$$y_{j_z} = \cos\left(\frac{2j_z - 1}{2n_z}\pi\right) \text{ and } \omega_{j_z} = \pi/n_z, \quad j_z = 1, \ldots, m_z.$$

Hence, the integral in (A.5) is written as

$$\sum_{j_k, j_i, j_h, j_\eta} \mathcal{R}(y_{j_k}, y_{j_l}, y_{j_h}, y_{j_\eta}; \mathbf{c}^{\tau+1})T_{i_k}(y_{j_k}T_{i_l}(y_{j_l})T_{i_h}(y_{j_h})T_{i_\eta}(y_{j_\eta}) = 0 \quad \text{(A.6)}$$

for $i_z = 1, \ldots, n_z, z \in \{k, l, h, \eta\}, i_z \in \Upsilon$. Define the product of the Chebyshev polynomials for $z \in \{k, l, h, \eta\}$ evaluated at the Chebyshev nodes $(y_{j_k}, y_{j_l}, y_{j_h}, y_{j_\eta})$ as $\mathbf{T}_i(y_j) \equiv T_{i,k}(y_{j_k})T_{i,l}(y_{j_l})T_{i,h}(y_{j_h})T_{i,\eta}(y_{j_\eta})$, and define $\mathbf{J}$ as the updated solution of the Bellman equation in (A.5) as

$$\mathcal{J}(\mathbf{c}^\tau) = \begin{bmatrix} \hat{J}^*(y_{1_k}, y_{1_l}, y_{1_h}, y_{1_\eta}; \mathbf{c}^\tau) \\ \vdots \\ \hat{J}^*(y_{m_k}, y_{m_l}, y_{m_h}, y_{m_\eta}; \mathbf{c}^\tau). \end{bmatrix}.$$

We will write these conditions in matrix form as

$$\mathcal{T} = \left[\begin{array}{c} \mathbf{T}_0(y_1) \ldots \mathbf{T}_0(y_{m_z}) \\ \vdots \ldots \vdots \\ \mathbf{T}_{n_z}(y_1) \ldots \mathbf{T}_{n_z}(y_{m_z}) \end{array} \right].$$

Using these definitions, we can write the orthogonality conditions in (A.5) in matrix form as $\mathcal{T}\mathcal{T}'\mathbf{c}^{\tau+1} = \mathcal{T}\mathcal{J}(\mathbf{c}^{\tau})$, which implies a new estimate of the Chebyshev coefficients $\mathbf{c}^{\tau+1}$ as a function of the coefficients $\mathbf{c}^{\tau}$ as

$$\mathbf{c}^{\tau+1} = \left(\mathcal{T}\mathcal{T}'\right)^{-1} \mathcal{T}\mathcal{J}(\mathbf{c}^{\tau}). \tag{A.7}$$

B Simulation-based Bayesian inference of the TFP processes

Given the underlying model described in the text, we assume that the agent cannot infer on the true data generating process (DGP) but she assumes that it can be either a model with low persistence or a high persistence. Since identification is the major challenge and plays a central role in our model and in agents' behavior, we use simulation-based Bayesian inference with noninformative priors to estimate the model parameters. Specifically, we use Gibbs sampling together with data augmentation (see Geman and Geman, 1984; Tanner and Wong, 1987) to obtain posterior results. Since the model is a special case of the unobserved components model with Gaussian distributions, we use the Kalman filter together with a simulation smoother. For the simulation smoother, we use the smoother proposed in Carter and Kohn (1994) and Frühwirth-Schnatter (1994). The resulting simulation scheme at the m^{th} step is as follows.

1. Sample ρ from $p(\rho|x_{0:T}^{(m-1)}, \sigma_y^2, \sigma_x^2, \bar{g})$.
2. Sample σ_y^2 from $p(\rho|x_{0:T}^{(m-1)}, \rho^{(m)}, \sigma_x^{2,(m-1)}, \bar{g}^{(m-1)})$.
3. Sample σ_x^2 from $p(\rho|x_{0:T}^{(m-1)}, \rho^{(m)}, \sigma_y^{2,(m)}, \bar{g}^{(m-1)})$.
4. Sample $\bar{g}$ from $p(\rho|x_{0:T}^{(m-1)}, \rho^{(m)}, \sigma_x^{2,(m)}, \sigma_y^{2,(m)})$.

5. Sample $x_{0:T}$, from $p(x_{0:T}|\rho^{(m)}, \sigma_y^{2,(m)}, \sigma_x^{2,(m)}, \bar{g}^{(m)})$ using Kalman filter and a simulation smoother.

While using noninformative priors reduces the posterior results to be identical with a pure likelihood based inference, it also provides us with the entire distribution of the model parameters. Examining the distribution of, most notably, the persistence parameter is an integral part of the approach followed in this paper in the sense that it provides a rational for agent's ignorance regarding the persistence of the TFP growth process. To evaluate the model, we use both the maximum likelihood and marginal likelihood values. The marginal likelihood can be computed as

$$p(y_{1:T}|M) = \int_\theta p(y_{1:T}|\theta)p(\theta)d\theta \tag{B.1}$$

where $\theta = (\rho, \sigma_y^2, \sigma_x^2, \bar{g})$. As the marginal likelihood is computed by integrating out the (prior) parameter distributions, it provides a robust way of computing a performance measure of the model. To compute the integral we use the modified harmonic mean estimator of Geweke (1999).

B.1 Informative priors

To have a more refined view of the impact of information and learning, we also generate estimates of the TFP process using more informative priors to improve the signal-to-noise ratio. We may view more informative prior distributions as a measure of the agent's "confidence" regarding the nature of the underlying TFP process as the informative content of the transitory component has increased. We assume a conjugate prior of an inverted Gamma function with two parameters as degrees of freedom (dof) and the scale. The results of the estimation are displayed in Table B.1. The results in Panel A through Panel F of this table are generated by assuming progressively more informative priors, and therefore an increasing signal-to-noise ratio compared to the uninformative prior case underlying the results in Table 5 in the text.

B.2 Industry estimates

In this section we present posterior results for 10 representative industries based on the data set from Bloom *et al.* (2018). The data are from the replication file provided by the Bloom *et al.*

Table B.1: Posterior results for the model using TFP for different values of ρ using the sample of 1947-2 : 1977-4.

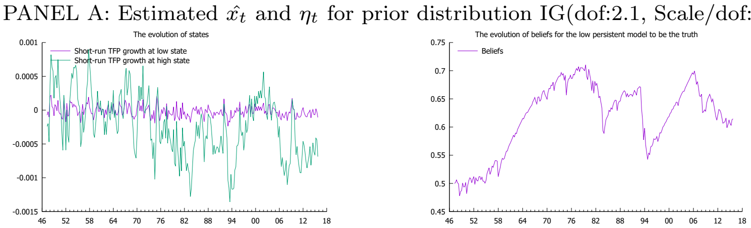
	Prior	Posterior	
		ρ: 0.300	ρ: 0.850
PANEL A			
$\bar{g}$	–	0.466 (0.083)	0.466 (0.084)
σ_A	IG(dof:2.1, Scale/dof:0.01)	0.925 (0.072)	0.940 (0.062)
σ_x	IG(dof:2.1, Scale/dof:0.01)	0.159 (0.109)	0.100 (0.037)
Maximum Likelihood		-167.40	-167.52
Marginal Likelihood		-170.08	-170.55
PANEL B			
$\bar{g}$	–	0.466 (0.083)	0.466 (0.081)
σ_A	IG(dof:2.1, Scale/dof:0.03)	0.905 (0.081)	0.936 (0.062)
σ_x	IG(dof:2.1, Scale/dof:0.03)	0.237 (0.127)	0.139 (0.037)
Maximum Likelihood		-167.41	-167.71
Marginal Likelihood		-170.23	-171.39
PANEL C			
$\bar{g}$	–	0.466 (0.078)	0.466 (0.084)
σ_A	IG(dof:2.1, Scale/dof:0.05)	0.891 (0.089)	0.933 (0.063)
σ_x	IG(dof:2.1, Scale/dof:0.05)	0.282 (0.135)	0.162 (0.044)
Maximum Likelihood		-167.42	-167.86
Marginal Likelihood		-170.34	-172.02
PANEL D			
$\bar{g}$	–	0.466 (0.078)	0.466 (0.083)
σ_A	IG(dof:2.1, Scale/dof:0.10)	0.867 (0.098)	0.928 (0.064)
σ_x	IG(dof:2.1, Scale/dof:0.10)	0.349 (0.139)	0.201 (0.049)
Maximum Likelihood		-167.43	-168.16
Marginal Likelihood		-170.50	-173.10
PANEL E			
$\bar{g}$	–	0.467 (0.074)	0.466 (0.082)
σ_A	IG(dof:2.1, Scale/dof:0.30)	0.819 (0.105)	0.915 (0.067)
σ_x	IG(dof:2.1, Scale/dof:0.30)	0.467 (0.134)	0.283 (0.057)
Maximum Likelihood		-167.51	-169.08
Marginal Likelihood		-170.67	-175.54
PANEL F			
$\bar{g}$	–	0.467 (0.072)	0.466 (0.082)
σ_A	IG(dof:2.1, Scale/dof:0.50)	0.796 (0.105)	0.907 (0.068)
σ_x	IG(dof:2.1, Scale/dof:0.50)	0.522 (0.126)	0.333 (0.061)
Maximum Likelihood		-167.56	-169.79
Marginal Likelihood		-170.78	-177.06

Note: The results show the posterior means and posterior standard deviations (in parenthesis) of the model parameters in the state space model displayed by equation (2.1-2.2) as burn-in sample. We kept every 5^{th} draw, which yields a sample of 10,000 draws from the ergodic distribution. IG stands for the inverted Gamma distribution with the parameters of degrees of freedom (dof) and scale.

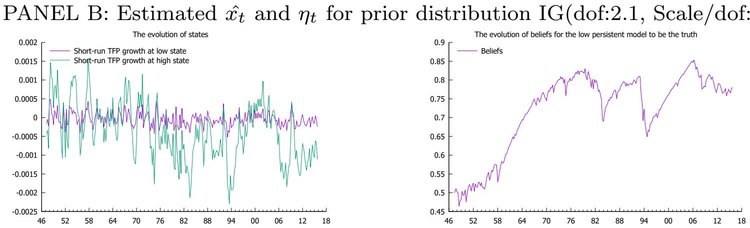
(2018). The data span 1971-2009 with annual frequency, and the estimates are based on TFP data expressed in percent. Hence, the parameters (except the ρ's) are divided by 100×4 in the computation of the agent's decision rule at the quarterly frequency. Below are the results for selected 4-digit SIC industries from group 2-digit SIC groups 24 Lumber and Wood Products, except Furniture, 28

Figure B.1: Evolution of the agent's beliefs for the low persistence model to be the true DGP over the sample 1978-1:2015-4.

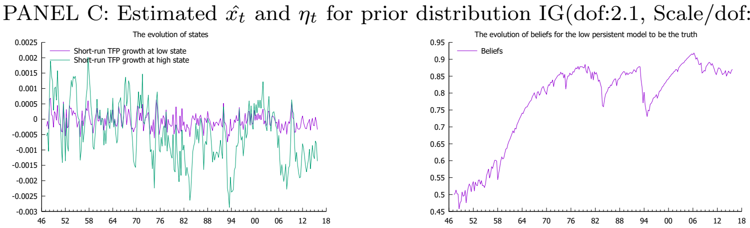
PANEL A: Estimated $\hat{x}_t$ and η_t for prior distribution IG(dof:2.1, Scale/dof:0.01)



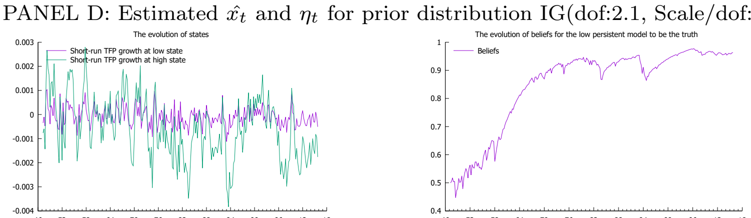
PANEL B: Estimated $\hat{x}_t$ and η_t for prior distribution IG(dof:2.1, Scale/dof:0.03)



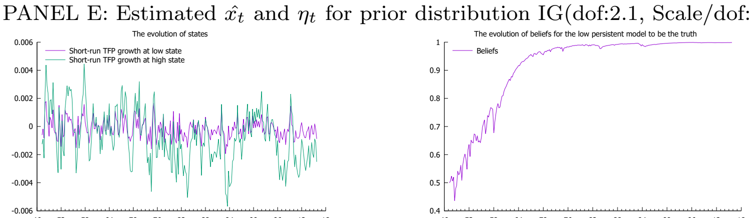
PANEL C: Estimated $\hat{x}_t$ and η_t for prior distribution IG(dof:2.1, Scale/dof:0.05)



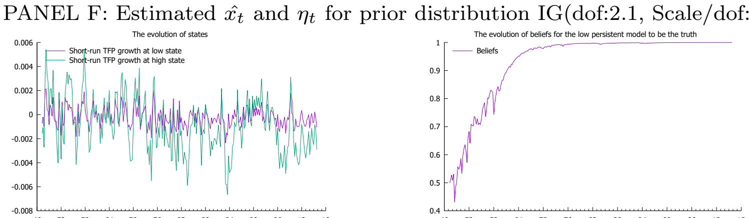
PANEL D: Estimated $\hat{x}_t$ and η_t for prior distribution IG(dof:2.1, Scale/dof:0.10)



PANEL E: Estimated $\hat{x}_t$ and η_t for prior distribution IG(dof:2.1, Scale/dof:0.30)



PANEL F: Estimated $\hat{x}_t$ and η_t for prior distribution IG(dof:2.1, Scale/dof:0.50)



Chemicals and Allied Products, 29 Petroleum Refining and Related Industries, 30 Rubber and Miscellaneous Products, 32 Stone, Clay, Glass and Concrete Products, 33 Primary Metal Industries, 34 Fabricated Metal Products (except Machinery and Transportation Equipment), 35 Industrial and Commercial Machinery and Computer Equipment, 36 Electronic and Other Electrical Equipment and Components (except Computer Equipment), and 37 Transportation Equipment. Our calculations are limited by the existence of sectoral data for which there exist significant missing observations throughout the entire sample period. Specifically, we dropped industries with more than 5 missing observations. We could not get the decision rules to either converge or yield plausible values for the unconditional moments under the TFP estimates for some industries, which have very high variances for the shocks $\epsilon_{A_k,t}$ and $\epsilon_{x_k,t}$ for $k = l, h$. Hence, we dropped those industries from consideration.

References

Altug, S. (1989), Time-to-build and Aggregate Fluctuations: Some New Evidence, *International Economic Review*, 30, 889–920.

Altug, S. and P. Labadie (2008), *Asset Pricing for Dynamic Economies*, Cambridge University Press.

Anderson, B. and J. Moore (1979), *Optimal Filtering*, New Jersey: Prentice Hall.

Angeletos, G. and J. La'O (2013), Sentiments, *Econometrica*, 81, 739–779.

Backus, D., A. Ferriere, and S. Zin (2015), Risk and ambiguity in models of business cycles, *Journal of Monetary Economics*, 69, 42–63.

Bansal, R. and A. Yaron (2004), Risks for the Long-run: A Poetntial Resolution of Asset Pricing Puzzles, *Journal fo Finance*, 59, 1481–1509.

Basu, S. and J. Fernald (1997), Returns to Scale in U.S. Production: Estimates and Implications, *Journal of Political Economy*, 105, 249–283.

Table B.2: Posterior results for the model using data on sectoral TFP for selected industries using the sample of 1947-2:1977-4

	2421: Sawmills and Planing Mills, General		2869: Industrial Organic Chemicals		2911: Petroleum Mining	
	0.30	0.85	0.30	0.85	0.30	0.85
$\bar{g}$	0.718 (0.083)	0.719 (0.098)	0.517 (0.086)	0.517 (0.096)	-0.449 (0.095)	-0.449 (0.099)
σ_A	1.767 (1.233)	2.978 (0.503)	1.589 (0.864)	2.273 (0.328)	4.830 (1.826)	5.667 (0.717)
σ_x	2.252 (1.179)	0.976 (0.623)	1.346 (0.888)	0.513 (0.379)	2.033 (2.066)	0.808 (0.694)
Max. Like.	-98.35	-98.74	-83.93	-84.08	-123.24	-123.24
Mar. Like.	-100.12	-101.64	-85.46	-86.36	-125.16	-125.22

	3084: Plastic Pipes		3315: Steel Wiredrawing and Steel Nails and Spikes		3448: Prefabricated Metal Buildings and Components	
	0.30	0.85	0.30	0.85	0.30	0.85
$\bar{g}$	-0.100 (0.100)	-0.100 (0.100)	-0.495 (0.079)	-0.494 (0.096)	0.547 (0.087)	0.545 (0.099)
σ_A	6.140 (0.899)	6.292 (0.761)	2.028 (1.916)	4.141 (1.558)	2.925 (2.005)	4.879 (0.741)
σ_x	1.130 (1.254)	0.692 (0.587)	3.805 (1.753)	1.851 (1.749)	3.035 (1.973)	0.833 (0.848)
Max. Like.	-123.35	-123.36	-116.87	-118.45	-111.42	-111.67
Mar. Like.	-125.08	-126.00	-118.71	-120.77	-112.80	-114.17

	3519: Internal combustion engines		3621: Motors and generators		3731: Ship building and repairing	
	0.30	0.85	0.30	0.85	0.30	0.85
$\bar{g}$	-0.536 (0.093)	-0.536 (0.100)	-0.259 (0.090)	-0.259 (0.099)	0.142 (0.099)	0.142 (0.099)
σ_A	3.919 (1.841)	5.148 (0.662)	2.680 (1.489)	3.907 (0.580)	5.179 (0.867)	5.345 (0.665)
σ_x	2.351 (1.991)	0.730 (0.705)	2.201 (1.555)	0.714 (0.702)	1.000 (1.1156)	0.630 (0.519)
Max. Like.	-119.14	-119.15	-108.93	-109.09	-113.97	-113.97
Mar. Like.	-120.91	-121.93	-110.56	-112.48	-115.80	-116.16

	3221: Glass Containers	
	0.30	0.85
$\bar{g}$	0.042 (0.095)	0.042 (0.098)
σ_A	2.489 (0.697)	2.763 (0.356)
σ_x	1.011 (0.872)	0.489 (0.356)
Max.	-93.10	-93.09
Mar.	-95.09	-95.56

Note: The results show the posterior means and posterior standard deviations (in parenthesis) of the model parameters the state space described by equations (2.1-2.2) in the text (evaluated in percentage terms). The inference was carried out with 60,000 draws where the first 10,000 are used as burn-in sample. We kept every 5^th draw, which yields a sample of 10,000 draws from the ergodic distribution. Max. Like. stands for the (logarithm of the) maximum likelihood whereas Mar. Like. stands for the (logarithm of the) marginal likelihood.

Beadry, P. and F. Portier (2006), Stock Prices, News, and Economic Fluctuations, *American Economic Review*, 96, 1293–1307.

Bianchi, F., C. L. Ilut, and M. Schneider (2017), Uncertainty Shocks, Asset Supply and Pricing over the Business Cycle, *The Review of Economic Studies*, 85, 810–854.

Bidder, R. and M. Smith (2012), Robust animal spirits, *Journal of Monetary Economics*, 59, 738–

750.

Bloom, N., M. Floetotto, N. Jaimovich, I. Saporta-Eksten, and S. J. Terry (2018), Really Uncertain Business Cycles, *Econometrica*, 86, 1031–1065.

Burnside, C. (1996), Production Function Regressions, Returns to Scale, and Externalities, *Journal of Monetary Economics*, 37, 177–201.

Cagetti, M., L. Hansen, T. Sargent, and S. Williamson (2002), Robustness and Pricing with Uncertain Growth, *Review of Financial Studies*, 15, 363–404.

Campbell, J. Y. (1996), Understanding Risk and Return, *Journal of Political Economy*, 104, 298–345.

Carter, C. K. and R. Kohn (1994), On Gibbs Sampling for State Space Models, *Biometrika*, 81, 541–553.

Christiano, L. and M. Eichenabum (1992), Current Real Business Cycle Theories and Aggregate Labor Market Fluctuations, *American Economic Review*, 82, 430–450.

Collard, F., S. Mukerji, K. Sheppard, and J.-M. Tallon (2018), Ambiguity and the historical equity premium, *Quantitative Economics*, 9, 945–993.

Collin-Dufresne, P., M. Johannes, and L. A. Lochstoer (2016), Parameter Learning in General Equilibrium: The Asset Pricing Implications, *American Economic Review*, 106, 664–98.

Croce, M. M. (2014), Long-run productivity risk: A new hope for production-based asset pricing?, *Journal of Monetary Economics*, 66, 13 – 31.

Demers, F., M. Demers, and S. Altug (2003), Investment Dynamics, in S. Altug, J. Chadha, and C. Nolan (eds.), *Dynamic Macroeconomic Analysis: Theory and Policy in General Equilibrium*, Cambridge University Press, pp. 34–154.

Fernald, J. G. (2012), A Quarterly, Utilization-adjusted Series on Total Factor Productivity, Working Paper Series 2012-19, Federal Reserve Bank of San Francisco.

Frühwirth-Schnatter, S. (1994), Data Augmentation and Dynamic Linear Models, *Journal of Time Series Analysis*, 15, 183–202.

Gallant, R., M. Jahan-Parvar, and H. Liu (2015), Measuring Ambiguity Aversion, Working Paper 2015-105, Pennsylvania State University.

Geman, S. and D. Geman (1984), Stochastic Relaxations, Gibbs Distributions, and the Bayesian Restoration of Images, *IEEE Transaction on Pattern Analysis and Machine Intelligence*, 6, 721–741.

Geweke, J. (1999), Using Simulation Methods for Bayesian Econometric Models: Inference, Development, and Communication, *Econometric reviews*, 18, 1–73.

Gilboa, I. and M. Schmeidler (1989), Maxmin Expected Utility with a Non-unique Prior, *Journal of Mathematical Economics*, 18, 141–153.

Gilchrist, S., J. Sim, and E. Zakrajsek (2014), Uncertainty, Financial Frictions, and Investment Dynamics, Working paper, Finance and Economics Discussion Paper, Federal Reserve Board.

Gordon, R. (2015), Secular Stagnation: A Supply-Side View, *American Economic Review: Papers and Proceedings*, 105, 54–59.

Hall, R. (1988), The Relationship between Price and Marginal Cost in US Industry, *Journal of Political Economy*, 96, 921–947.

Hall, R. (1990), Invariance Properties of Solow's Productivity Residual, in P. Diamond (ed.), *Growth, Productivity and Employment*, The MIT Press, pp. 71–112.

Hansen, G. (1985), Indivisible Labor and the Business Cycle, *Journal of Monetary Economics*, 16, 309–327.

Hansen, L. (2007), Prior Beliefs and Doubts, *American Economic Review*, 97, 1–30.

Hansen, L. and T. J. Sargent (2008), *Robustness*, Pinceton University Press.

Ilut, C. and M. Schneider (2014), Ambiguous Business Cycles, *American Economic Review*, 104, 2368–99.

Jahan-Parvar, M. R. and H. Liu (2014), Ambiguity aversion and asset prices in production economies, *The Review of Financial Studies*, 27, 3060–3097.

Jaimovich, N. and S. Rebelo (2009), Can News about the Future Drive the Business Cycle?, *American Economic Review*, 99, 1097–1118.

Ju, N. and J. Miao (2007), Ambiguity, Learning, and Asset Returns, Working paper, Boston University.

Ju, N. and J. Miao (2012), Ambiguity, Learning, and Asset Returns, *Econometrica*, 80, 559–591.

Judd, K. (1998), *Numerical Methods in Economics*, The MIT Press.

Kaltenbrunner, G. and L. Lochstoer (2010), Long-run Risk through Consumption Smoothing, *Review of Financial Studies*, 23, 3190–3224.

Klibanoff, P., M. Marinacci, and S. Mukerji (2005), A Smooth Model of Decision Making under Ambiguity, *Econometrica*, 73, 1849–1892.

Klibanoff, P., M. Marinacci, and S. Mukerji (2009), Recursive Ambiguity Preferences, *Journal of Economic Theory*, 144, 930–976.

Liu, H. and Y. Zhang (2018), Financial Uncertainty with Ambiguity and Learning, Working paper, University of Manchester.

Nimark, K. (2014), Man-Bites-Dog Business Cycles, *American Economic Review*, 104, 2320–2367.

Prescott, E. C. (1986), Theory Ahead of Business Cycle Measurement, *Federal Reserve Bank of Minneapolis Quarterly Review*, 10, 9–21.

Richter, A. W., N. A. Throckmorton, and T. B. Walker (2014), Accuracy, speed and robustness of policy function iteration, *Computational Economics*, 44, 445–476.

Stock, J. and M. Watson (2002), Has the Business Cycle Changed and Why?, in M. Gertler and K. Rogoff (eds.), *NBER Macroeconomics Annual*, National Bureau of Economic Research.

Strzalecki, T. (2013), Temporal Resolution of Uncertainty and Recursive Models of Ambiguity Aversion, *Econometrica*, 81, 1039–1074.

Tallarini, T. (2000), Risk-sensitive Real Business Cycles, *Journal of Monetary Economics*, 45, 507–552.

Tanner, M. A. and W. H. Wong (1987), The Calculation of Posterior Distributions by Data Augmentation, *Journal of the American Statistical Association*, 82, 528–550.

Vissing-Jørgensen, A. and O. P. Attanasio (2003), Stock-Market Participation, Intertemporal Substitution, and Risk-Aversion, *American Economic Review*, 93, 383–391.