Solving Dynamic Heterogeneous Agent Models: Taking Stock

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Overview

- Recent two decades has seen vast advances on solving dynamic heterogeneous agent models.
  - Many agents with iid shocks ("employed", "unemployed").
  - They self-insure by holding capital.
  - Aggregate productivity fluctuations.
  - Date zero may not be steady state.
  - Thus, dynamics: returns and wages fluctuate over time.
- Many approaches.
- Clever, ingenious, but also daunting literature.
- My goal today: taking stock. Overview, starting from the basics.
- So: ease the entry. Not: "rat race". Not: code repository.
- Broad brush. Ignore many subtleties. Still: lots of equations.
- Steady state? Not today. I assume we know it.
- Apologies for what I left out and all mistakes. Comments welcome.

Taylor - Uhlig (1990), Judd book (1998), Heer - Maußner book, 3rd ed (2024)

Basics: rep agent neoclass growth model, discrete time

- Households: fix labor $\bar{n}$.
- Given k_0, z_0, wages w_t, rental rates r_t, solve

$$v(k_0, z_0) = \max_{(c_t, k_{t+1})_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right] \text{ s.t.} \tag{1}$$

$$c_t + k_{t+1} = w_t \bar{n} + (1 + r_t - \delta) k_t \tag{2}$$

with c_t and k_{t+1} chosen at t (or: adapted to $\mathcal{F}_t$).

- Firms: given wages w_t, rental rates r_t, solve

$$\max_{n,k} y_t - w_t n - r_t k \text{ s.t.} \tag{3}$$

$$y_t = e^{z_t} k^{\theta} n^{1-\theta} \tag{4}$$

- Equilibrium: Market clearing.

$$\bar{n} = n, \ k_t = k \tag{5}$$

- Stochastics:

$$z_{t+1} = \rho z_t + \epsilon_t, \ \sigma \epsilon_{t+1} \overset{i.i.d}{\sim} \mathcal{N}(0, 1) \tag{6}$$

Dyn Prog and FONCs

- Dynamic programming. Household $\mathbf{i} \in [0,1]$ with $k_{\mathbf{i}}$:

$$v(k_{\mathbf{i}}; \underbrace{k, z}_{S}) = \max_{c_{\mathbf{i}}, k'_{\mathbf{i}}} u(c_{\mathbf{i}}) + \beta E[v(k'_{\mathbf{i}}; \underbrace{k', z'}_{S'})], \quad \text{s.t.}$$
$$c_{\mathbf{i}} + k'_{\mathbf{i}} = w(S)\bar{n} + (1 + r(S) - \delta)k_{\mathbf{i}}$$
$$(k', z') = g(k, z, \epsilon')$$

- Consistency: $k_{\mathbf{i}} = k$ implies $c_{\mathbf{i}} = c$ and $k'_{\mathbf{i}} = k$
- Households:

$$1 = \beta E_t \left[\frac{u'(c_{t+1})}{u'(c_t)} (1 + r_{t+1} - \delta) \right] \tag{7}$$

where the expectation is taken over the aggregate shock ϵ_{t+1}.

- Firms:

$$w_t = (1 - \theta)\frac{y_t}{n_t}, \text{ and } r_t = \theta \frac{y_t}{k_t} \tag{8}$$

Solutions

- **State**: $\mathbf{S_t} = (k_t, z_t) \in \mathbb{R}^2$. Endogenous part: k_t. Exogenous part: z_t.
- **Controls**: everything else. $\mathbf{C_t} = (c_t, y_t, w_t, r_t)$.
- What is a solution? Sequence space or state space:
  - A **stochastic sequence** $(S_t, C_t, \epsilon_t)_{t=0}^{\infty}$
  - or a **recursive law of motion (RLOM)**:
    $(C_t, S_{t+1}) = (\mathbf{f}(S_t), \mathbf{g}(S_t, \epsilon_{t+1}))$

  such that equations (2,4,5,6,7,8) hold. Or:
- **Kolmogorov forward equation**: $S_{t+1} = g(S_t, \epsilon_{t+1})$.
- Equations, summarized:

$$0 = E_t[\mathbf{F}(S_t, C_t, S_{t+1}, C_{t+1}, \epsilon_{t+1})] \tag{9}$$

- Numerical solutions: approximate f or g. Or sequence.
- Numerical simulation ("Pruning").

Schmitt-Grohé - Uribe (2004), Bayer - Luetticke (2020), KKSS (2008)

Sequence space approach

- The problem, abstract:

$$\text{Solve} \quad 0 \quad = E_t[F(\mathbf{S_t}, \mathbf{C_t}, \mathbf{S_{t+1}}, \mathbf{C_{t+1}}, \epsilon_{t+1})]$$

  for $(\mathbf{C_t}, \mathbf{S_{t+1}})_{t=0}^{T}$so that the solution is stable.
- E.g. **terminal condition**: $S_{T+1} = \bar{S}$ and $\epsilon_{T+1} = 0$.
- **System of equations** $t = 0, \ldots, T$.
- Perturbation approach: Taylor expansion around steady state.
- Linearization. Without shocks, after canceling constants:

$$0 = F_S S_t + F_C C_t + F_{S'} S_{t+1} + F_{C'} C_{t+1},\ t = 0, \ldots, T$$

  or, with $S_{t+1} = L^{-1} S_t$ and $\mathbf{S} = [S_t]_{t=0}^{T}$, $\mathbf{C} = [C_t]_{t=0}^{T}$,

$$0 = [F_S,\ F_C,\ F_{S'},\ F_{C'}] \begin{bmatrix} I & 0 \\ 0 & I \\ L^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{S} \\ \mathbf{C} \end{bmatrix}$$

- Solve for **S** and **C**: matrix algebra. Impulse responses.

Judd book (1998)

State space approach

- The problem, abstract:

$$\text{Solve} \quad 0 \quad = E_t[F(S, C, S', C', \epsilon')]$$

  for $(C, S') = (\mathbf{f}(S), \mathbf{g}(S, \epsilon'))$, so that the solution is stable.
- Functional equation:

$$0 = E_{\epsilon'}[F(S, \mathbf{f}(S), \mathbf{g}(S, \epsilon), \mathbf{f}(\mathbf{g}(S, \epsilon)), \epsilon')]$$

Perturbations

- Functional equation:

$$\begin{aligned} 0 &= E_t[F(S, \ C, \ \ S', \ \ C', \ \ \epsilon')] \\ \text{becomes } 0 &= E_{\epsilon'}[F(S, \mathbf{f}(S), \mathbf{g}(S,\epsilon), \mathbf{f}(\mathbf{g}(S,\epsilon)), \epsilon')] \end{aligned}$$

- Linearize around steady state. After canceling constants:

$$\begin{aligned} 0 &= F_S + F_C \mathbf{f_S} + F_{S'} \mathbf{g_S} + F_{C'} \mathbf{f_S} \mathbf{g_S} \\ &= [F_S \; F_C] \begin{bmatrix} 1 \\ \mathbf{f_S} \end{bmatrix} + [F_{S'} \; F_{C'}] \begin{bmatrix} 1 \\ \mathbf{f_S} \end{bmatrix} \mathbf{g_S} \end{aligned}$$

- Compare to generalized eigenvalue problem, Λ_{11} stable, Λ_{22} unstable:

$$0 = [F_S \; F_C] \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} + [F_{S'} \; F_{C'}] \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix}$$

$$\text{Thus: } 0 = [F_S \; F_C] \begin{bmatrix} 1 \\ Z_{21} Z_{11}^{-1} \end{bmatrix} + [F_{S'} \; F_{C'}] \begin{bmatrix} 1 \\ Z_{21} Z_{11}^{-1} \end{bmatrix} \quad Z_{11} \Lambda_{11} Z_{11}^{-1}$$

- Solution: $\mathbf{f_S} = Z_{21} Z_{11}^{-1}$ and $\mathbf{g_S} = Z_{11} \Lambda_{11} Z_{11}^{-1}$.
- Higher-order pertubations: no root selection, just linear equations.

Uhlig (2001), Sims (2001), Schmitt-Grohé - Uribe (2004), Judd book (1998)

The rep agent neoclass growth model: discrete time

- Households: fix labor $\bar{n}$.
- Given (k_0, z_0), wages w_t, rental rates r_t, solve

$$v(\underbrace{k_0, z_0}_{S_0}) = \max_{(c_t, k_{t+1})_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t\, u(c_t)\right] \text{ s.t.} \tag{10}$$

$$c_t + k_{t+1} = w_t \bar{n} + (1 + r_t - \delta) k_t \tag{11}$$

- Firms: given w_t, r_t, solve $\max_{n,k} y_t - w_t n - r_t k$ s.t.

$$y_t = e^{z_t} k^{\theta} n^{1-\theta} \tag{12}$$

- Equilibrium: Market clearing.

$$\bar{n} = n,\ k_t = k \tag{13}$$

- Stochastics:

$$z_{t+1} = \rho z_t + \epsilon_t,\ \sigma \epsilon_{t+1} \overset{i.i.d}{\sim} \mathcal{N}(0, 1) \tag{14}$$

The stochastic heterogeneous agents model

- Households: $i \in [0,1]$, **Markov process** $n_{i,t} \in \{n^{(1)}, \ldots, n^{(J)}\}$.
- Given $(\mathbf{k_{i,0}}, \mathbf{n_{i,0}})$,$(\mu_{\mathbf{0}}, z_0)$, wages w_t, rental rates r_t, HH **i** solves

$$v(\underbrace{k_{\mathbf{i},0}, \mathbf{n_{i,0}}}_{s_{\mathbf{i},0}}; \underbrace{\mu_{\mathbf{0}}, z_0}_{S_0}) = \max_{(c_t, k_{t+1})_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t u(c_{\mathbf{i},t})\right] \text{ s.t.} \tag{15}$$

$$c_{\mathbf{i},t} + k_{\mathbf{i},t+1} = w_t \mathbf{n_{i,t}} + (1 + r_t - \delta) k_{\mathbf{i},t} \tag{16}$$

- Firms: given w_t, r_t, solve $\max_{n,k} y_t - w_t n - r_t k$ s.t.

$$y_t = e^{z_t} k^{\theta} n^{1-\theta} \tag{17}$$

- Equilibrium: Market clearing. With $\mathbf{s_{i,t}} = (\mathbf{k_{i,t}}, \mathbf{n_{i,t}})$,

$$\int n_{\mathbf{i},t} d\mathbf{i} = \int \mathbf{n}(\mathbf{s}, \mathbf{S_t}) \mathbf{d}\mu_{\mathbf{t}}(\mathbf{s}) = n, \int k_{\mathbf{i},t} d\mathbf{i} = \int \mathbf{k}(\mathbf{s}, \mathbf{S_t}) \mathbf{d}\mu_{\mathbf{t}}(\mathbf{s}) = k \tag{18}$$

- Stochastics: **idiosynchratic Markov processes for** $n_{\mathbf{i},t}$ and

$$z_{t+1} = \rho z_t + \epsilon_t, \ \sigma \epsilon_{t+1} \overset{i.i.d}{\sim} \mathcal{N}(0,1) \tag{19}$$

Uhlig thesis (1990), Hugget (1993), Aiyagari (1994), Krusell-Smith (1998)

FONCs

- Dynamic programming. Household $\mathbf{i} \in [0,1]$ with $s_{\mathbf{i}} = (k_{\mathbf{i}}, n_{\mathbf{i}})$:

$$v(\underbrace{k_{\mathbf{i}}, n_{\mathbf{i}}}_{s_{\mathbf{i}}}; \underbrace{\mu, z}_{S}) = \max_{c,k'} u(c) + \beta E[v(\underbrace{k'_{\mathbf{i}}, n'_{\mathbf{i}}}_{s'_{\mathbf{i}}}; \underbrace{\mu', z'}_{S'})], \quad \text{s.t.}$$

$$c + k'_{\mathbf{i}} = w(S)n_{\mathbf{i}} + (1 + r(S) - \delta)k_{\mathbf{i}}$$

$$S' = g(S, \epsilon')$$

$$n_{\mathbf{i}} \overset{\text{Markov}}{\rightarrow} n'_{\mathbf{i}}$$

- Consistency: Kolmogorov forward equation (next slide).
- Households:

$$1 = \beta E_{\mathbf{i},t}\left[\frac{u'(c_{\mathbf{i},t+1})}{u'(c_{\mathbf{i},t})}(1 + r_{t+1} - \delta)\right] \tag{20}$$

  where the expectation is taken
  - over the aggregate shock ϵ_{t+1}
  - and **the idiosyncratic Markov transition rates $n_{\mathbf{i},t} \rightarrow n_{\mathbf{i},t+1}$**.
- Firms:

$$w_t = (1-\theta)\frac{y_t}{n_t}, \text{ and } r_t = \theta\frac{y_t}{k_t} \tag{21}$$

Solutions

- Aggregate state: $S_t = (\mu_{\mathbf{t}}, z_t) \in \mathbb{R}^\infty$.
- **Idiosynchratic state** $\mathbf{s}_{\mathbf{i},\mathbf{t}} = (k_{i,t}, n_{i,t}) \in \mathbb{R}^2$. **Dec. rule** $\mathbf{k}(\mathbf{s}, \mathbf{S})$.
- Aggregate controls: (y_t, w_t, r_t).
- **Idiosynchratic controls**: $c_{\mathbf{i},t}$. **Dec. rule** $\mathbf{c}(\mathbf{s}, \mathbf{S})$.
- Controls: $C_t = (y_t, w_t, r_t, \mathbf{k}(\cdot,\cdot), \mathbf{c}(\cdot,\cdot))$
- What is a solution? Sequence space or state space:
  - A stochastic sequence $(S_t, C_t, \epsilon_t)_{t=0}^\infty$
  - or a RLOM: $(C_t, S_{t+1}) = (f(S_t), g(S_t, \epsilon_{t+1}))$

  such that
  - equations (16,17,18,19,20,21)
  - and **Kolmogorov forward equation** hold. For set $K \subset \mathcal{B}(\mathbb{R})$:

$$\mu_{t+1}(K, n^j) = \int_{k(s,S_t)\in K} P(n' = n^{(j)} \mid n = n(s))\, d\mu_t(s) \quad (22)$$

- Equations, summarized:

$$0 = E_{\mathbf{s},t}[F(S_t, C_t, S_{t+1}, C_{t+1}, \epsilon_{t+1})], \mathbf{s} \in \mathcal{S} \quad (23)$$

Bayer - Luetticke (2020)

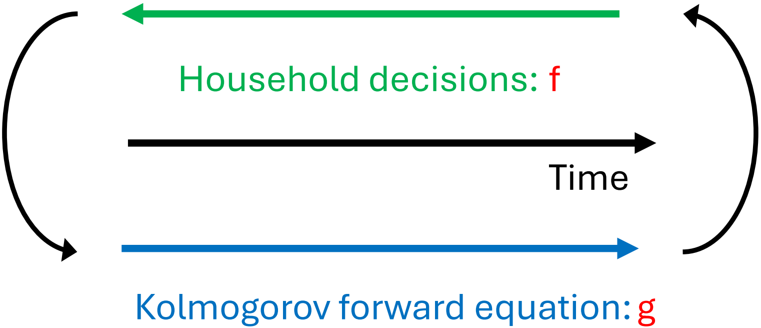
Sequence approach, state space approach

- Conceptually: as before.
- Three challenges:
  - The system of equations (23) is ∞-dimensional.
  - $S_t = (\mu_{\mathbf{t}}, z_t) \in \mathbb{R}^\infty$. Or better: some suitable vector space.
    - Linearize.
    - Finite-dimensional approximation.
  - What to linearize?
    - Not: $k(s_t, S_t) \approx \bar{k} + k_s(s_t - \bar{s}) + k_S(S_t - \bar{S})$
    - Rather: $k(\cdot, S_t) \approx k(\cdot, \bar{S}) + k_S(\cdot, S_t - \bar{S})$
    - Substantial nonlinearities are essential part of the solution.
    - $S_t \in \mathbb{R}^\infty$. So, $k_S : \mathbb{R} \times \mathbb{R}^\infty \to \mathbb{R}$.

Forward and backward: a fixed point problem

$$s = (k, n), \qquad S = (\mu, z)$$

$$v(s; S) = \max_{c,s'} \{u(c) + \beta E[v(s'; S' = g(S, \epsilon'))] \mid \text{“budg. constr.”}(s, S)\}$$



$$\int P(s' \in \mathcal{S} \mid s)\mu(ds) = \mu'(s' \in \mathcal{S})$$

May solve both simultaneously rather than iteratively. E.g. linearizing.

Krieger: Eigenvectors in $\mathbb{R}^\infty$.

- **Krieger, 1997**. Last revision: 2002. Working papers.
- Set $\epsilon_t = 0, t > 0$ ("Impulse response").
- Consider linear approximation to g. Canceling constants: $S' = g_S S$
- $g_S : \mathbb{R}^\infty \to \mathbb{R}^\infty$: a **linear operator** ("matrix").
- Take a few important eigenvalues $\lambda_1, \ldots, \lambda_M$.
- Eigenvectors $\mathbf{S^{(1)}}, \ldots, \mathbf{S^{(M)}}$.
- Evolution of aggregate state, if $S_0 = S^{(m)}$: $S_t = \lambda_m^t S^{(m)}$.
- With linear approximation, true for everything else:

$$r_t = r(S_t) \approx \lambda_m^t r_0$$

- If $S_0 = \sum_{m=1}^M \alpha_{m,0} S^{(m)}$, similar: $S_t = \sum_{m=1}^M \lambda_m^t \alpha_{m,0} S^{(m)}$.
- Approximate aggregate state S_t by weights $\alpha_t = [\alpha_{1,t}, \ldots, \alpha_{M,t}]'$.
- $\alpha_{t+1} = \Lambda \alpha_t$, where Λ diagonal, $\Lambda_{mm} = \lambda_m$.
- Advantage: everything moves in simple fashion:

$$r_t = r(S_t) \approx \sum_{m=1}^M \lambda_m^t \alpha_{m,0} r_0^{(m)} = \sum_{m=1}^M \alpha_{m,t} r_0^{(m)}$$

- With $(\lambda_m, S^{(m)})_{m=1}^M$ solve HH problem. Update $(\lambda_m, S^{(m)})_{m=1}^M$.

Krusell-Smith: Moments of μ_t

- **Krusell-Smith, JPE 1998**.
- Summarize μ_t by key moments $\mathbf{m}_t = [m_{1,t}, \ldots, m_{I,t}]'$.
- Mean, variance, ... In Krusell-Smith (1998), mean is enough.
- (Ignoring z_t-bit:) Approximate $\mu_{t+1} = g(\mu_t)$ with

$$\mathbf{m}_{t+1} = \tilde{g}(\mathbf{m}_t; \theta)$$

  where θ are parameters to be solved for.
- Example: $\theta = (\bar{\mathbf{m}}, A)$, where $\bar{\mathbf{m}}$ vector, A matrix, and

$$\log(\mathbf{m}_{t+1}) = \log(\bar{\mathbf{m}}) + A \log(\mathbf{m}_t)$$

- Given θ and $S \approx (\mathbf{m}, z)$, agents can forecast $S' \approx (\mathbf{m}', z')$.
- Simulate N HH's. Update θ by fitting to resulting evolution of μ_t.
- **Algan, Allais and Haan, JEDC 2008**, **Winberry, QE 2018**:

$$\text{dens}(\mu_t)(k, n^{(j)}) = \alpha_{\mathbf{0,j}} e^{\alpha_{\mathbf{1,j}}(\log k - \mathbf{m_{1,j}}) + \sum_{i=2}^{I} \alpha_{\mathbf{i,j}}((\log k - \mathbf{m_{1,j}})^i - \mathbf{m_{i,j}})}$$

  Key: calculate $\alpha_{\mathbf{i,j}}'s$ from moments $\mathbf{m_{i,j}}'s$. Projection, not simulation.

Reiter: histogram, linearization of g

- **Reiter, JEDC 2009**
- Select grid points $0 = k^{(0)}, k^{(1)}, \ldots, k^{(M)} = \bar{k}$.
- Approximate μ_t by histogram, i.e. the probabilities

$$p_{t,m,j} = \mu_t((k,n) \in (k^{(m)}, k^{(m+1)}] \times \{n^{(j)}\})$$

- $\mathbf{p}_t = [p_{t,m,j}]_{m,j}$ is an JM-dimensional vector.
- (Ignoring z_t-bit:) Approximate $\mu_{t+1} = g(\mu_t)$ with

$$\mathbf{p}_{t+1} = \mathbf{\Pi}\mathbf{p}_t$$

  for some $JM \times JM$-matrix $\mathbf{\Pi}$ to be solved for.
- Given Π, solve agent problem. Update Π.
- Further dimens. reduction: project $\mathbf{p}_t - \bar{p}$ on $L < JM$ vectors b_l:

$$\mathbf{p}_t - \bar{p} \approx B\mathbf{a}_t, \text{ where } B = [b_1, \ldots, b_L]$$

  Solve for $L \times L$ matrix $\mathbf{A}$, where $\mathbf{a}_{t+1} = \mathbf{A}\mathbf{a}_t$.
- **Bayer-Luetticke, QE 2020**: sparse expansions, “image compression”. Discrete cosine transformations.

Auclert et al: sequence space, linearization

- **Auclert-Bardóczy-Rognlie-Straub, Econometrica 2021**.
- Linearization of the sequence of equations.
- Impulse responses: so $\epsilon_t = 0, t > 0$.
- Given: $\mathbf{z} = (z_t)_{t=0}^{\infty}$. Find aggr capital sequence $\mathbf{K} = (K_t)_{t=0}^{T}$.
- Initial condition μ_0, z_0. Terminal condition: steady state at $T+1$.
- Firm's FOCs: $\mathbf{K}, \mathbf{z} \quad \rightarrow \quad (w_t, r_t)_{t=0}^{T}$
- HH"s: $(w_t, r_t)_{t=0}^{T} \quad \rightarrow \quad \mathbf{k}_t'(k, n)$
- Kolmogorov: $\mathbf{k}_t'(k, n) \quad \rightarrow \quad \mathbf{K}$
- Together: $\mathcal{K} : (\mathbf{K}, \mathbf{z}) \rightarrow \mathbf{K}$. Thus, $\mathbf{0} = \mathbf{H}(\mathbf{K}, \mathbf{z}) = \mathcal{K}(\mathbf{K}, \mathbf{z}) - \mathbf{K}$
- Implicit function theorem: $d\mathbf{K} = -\mathbf{H}_{\mathbf{K}}^{-1}\mathbf{H}_{\mathbf{z}}\, d\mathbf{z}$.
- Calculating these derivatives: tricky. They exploit lots of structure.
- "Fake news algorithm": date-s shock at $t = 0$, retracted at $t = 1$.
- **Bhandari-Bourany-Evans-Golosov, NBER 2023**: higher order.

Ahn et al: cont time, linearization

- **Ahn-Kaplan-Moll-Winberry-Wolf, NBER Macro-Annual 2018**
- value function on k-grid and $n^{(1)}, \ldots, n^{(J)}$: vector $\mathbf{v}_t$.
- dens(μ_t) on k-grid and $n^{(1)}, \ldots, n^{(J)}$: vector $\mathbf{h}_t$.
- prices: $\mathbf{p}_t = (w_t, r_t) = P(\mathbf{h}_t; z_t)$ per market clearing.
- HH optim. results in utility $\mathbf{u}(\mathbf{v_t})$, s' per $\mathbf{A}(\mathbf{v_t}; \mathbf{p_t})$. Four equations:

$$
\begin{aligned}
\text{Dyn Prog:} \quad \rho \mathbf{v}_t &= \mathbf{u}(\mathbf{v}_t) + \mathbf{A}(\mathbf{v}_t; \mathbf{p}_t)\mathbf{v}_t + \frac{E_t[d\mathbf{v}_t]}{dt} \\
\text{Kolmogorov:} \quad \tfrac{d\mathbf{h}_t}{dt} &= \mathbf{A}(\mathbf{v}_t; \mathbf{p}_t)^T \mathbf{h}_t \\
\text{Productivity:} \quad dz_t &= -\eta z_t dt + \sigma dW_t \\
\text{Prices:} \quad \mathbf{p}_t &= P(h; z_t)
\end{aligned}
$$

- Cont time: sparse $\mathbf{A}(\mathbf{v}_t; \mathbf{p}_t)$, since assets move little.
- Hats: ss - dev.. Let $\hat{\mathbf{x}}_t = [\hat{\mathbf{v}}_t, \hat{\mathbf{h}}_t, \hat{z}_t]'$. Linearization, subst. out $\hat{\mathbf{p}}_t$:

$$E_t[d\hat{\mathbf{x}}_t] = \mathbf{B}\ \hat{\mathbf{x}}_t dt$$

- Use cont. time version of "Perturbations": find stable roots, RLOM

$$
\begin{aligned}
\text{“}f'' : \quad \mathbf{v}_t &= \mathbf{D_{vg}} \mathbf{h}_t + \mathbf{D_{vz}} z_t \\
\text{“}g'' : \quad \tfrac{d\mathbf{h}_t}{dt} &= \mathbf{D_{hh}} \mathbf{h}_t + \mathbf{D_{hz}} z_t
\end{aligned}
$$

Bilal: cont time, master equation

- **Bilal, NBER 2023**
- **Mean field games: Lions (2011)**.
- Four equations, similar to Ahn-et-al. No grid.

$$\begin{aligned}
\text{Dyn Prog:} \quad \rho \mathbf{v}_t(s) &= \max_{\mathbf{c}} \mathbf{u}(c) + \mathbf{L_t(s,c)}[\mathbf{v}_t] + \frac{d\mathbf{v}_t}{dt}(s) && (24)\\
\text{Kolmogorov:} \quad \tfrac{d\mathbf{h}_t}{dt} &= \mathbf{L_t^*(s,c(s))}[\mathbf{h}_t] && (25)\\
\text{Productivity:} \quad dz_t &= -\eta z_t dt + \sigma dW_t && (26)\\
\text{Prices:} \quad (w_t, r_t) &= (\mathcal{W}(h_t, z_t), \mathcal{R}(h_t, z_t)) && (27)
\end{aligned}$$

- **State space:** $\mathbf{v}_t(s) = \mathbf{v}(s, h_t)$. So, $\frac{d\mathbf{v}_t}{dt}(s) = < \frac{\partial \mathbf{v}}{\partial h}(s, h_t), \frac{\partial h_t}{dt} >$.
- **Master equation**: replace $\frac{d\mathbf{v}_t}{dt}(s)$ in (24) per (25):

$$\rho \mathbf{v}(s;h) = \max_{\mathbf{c}} \mathbf{u}(c) + L(s,c,\mathbf{h})[\mathbf{v}] + \int \frac{\partial \mathbf{v}}{\partial h}(s,s',h) L^*(s',\mathbf{h})[h] ds' \quad (28)$$

- Everything is in one equation. In spirit, like $\mathcal{K} : (\mathbf{K}, \mathbf{z}) \to \mathbf{K}$, but for $\mathbf{v}$.
- **F**irst-order **A**pproximation to the **M**aster **E**quation: **FAME**.

Global Solutions, Deep Learning

- **Gu-Lauriére-Merkel-Payne, "Global Solutions...", WP 2024**
  - Compare three discretizations for h_t (or μ_t):
    - Simulate N agents: **robust**.
    - grid: **difficult to work with**.
    - projecting on finite set of basis functions: **low-dim., best**.
  - Approximate value function with **neural net**: $v(s; h; \theta)$.
  - Goal: solve the master equation in cont. time as in Bilal (2023).
  - Train to solve it, using **deep learning** tools:
    - Sample random points (x, h) in discretized state space.
    - Calculate master equation error.
    - Update θ.
  - **EMINN**: **E**conomic **M**odel **I**nformed **N**eural **N**etwork.
- **Han-Yang-E, "DeepHAM ...", WP 2021**
  - Discrete time, as in Krusell-Smith (1998).
  - Simulate N agents.
  - Approximate decision rules and value functions with neural nets.

Conclusions

- Recent two decades has seen vast advances on solving dynamic heterogeneous agent models.
- I took stock.
- Remarkably ingenious and diverse solution techniques.
- Key is **dimensionality reduction** $\mu_t \in \mathbb{R}^\infty$.
  - Moment techniques: **Krusell-Smith**, **Algan et al.**, **Winberry**.
  - Linearization. For agent distribution μ_t or h_t:
    - eigenvectors: **Krieger**.
    - histogram, perhaps projected: **Reiter**, **Bayer-Luetticke**.
    - continous time, grid: **Ahn et al**.
    - master equation, FAME: **Bilal**.
  - Linearization and sequence approach: **Auclert et al**.
  - Neural nets, deep learning: **Gu et al**, **Han et al**.
- Best papers, best practice:
  - start from encompassing framework and baseline example.
  - explain key parts clearly rather than magic black box.
  - show skeletons in the closet. Not just "best performance ever".
- I hope this was useful. **Thanks!**